Scientific Computing-Exercises

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Abstract

1 Convergence

1.1 Smooth Solution

Discretize the boundary value problem: Find u such that

$$\begin{aligned} -\varepsilon^2 u'' + u &= 0 \\ u(0) &= 0 \\ u(1) &= 1 \end{aligned}$$

using the p-Version of the FEM with one element and $p = 2, 3, \ldots, 20$. Determine the correspond-inmg system of linear algeraic equations which you can solve by a direct method.
What convergence speed do you obtain for different values of $\varepsilon = 1, 0.1, 0.01, 0.001$. Compare with the exact solution.

1.2 Smooth Solution 2

Consider the boundary value problem: Find u such that

$$\begin{aligned} -u'' + u &= 1 \\ u(-1) &= 0 \\ u(1) &= 0 \end{aligned}$$

- Compute the exact solution,
- Prove exponential convergence of p-FEM by using the exact solution and their representation

$$u(x) = \sum_{i=2}^{p} \alpha_i L_i(x) \quad \text{with} \quad \alpha_i = \frac{2i-1}{2} \int_{-1}^{1} u'(x) L_i'(x) \, \mathrm{d}x.$$

1.3 Singular solution

Discretize the boundary value problem: Find u such that

$$\begin{aligned} -u'' + u &= f \\ u(-1) &= 1 \\ u(1) &= 0 \end{aligned}$$

using the p-Version of the FEM with one element and $p = 2, 3, \ldots, 20$. f is chosen such that the exact solution is $u = \left(\frac{1}{2}\sqrt{2-2x}\right)^3$. Determine the correspondinmg system of linear algeraic equations which you can solve by a direct method. What is the convergence, in particular around $x = 1$.

1.4 Singular solution

Discretize the boundary value problem: Find u such that

$$\begin{aligned} -u'' + u &= f \\ u(-1) &= 1 \\ u(1) &= 0 \end{aligned}$$

using the p-Version of the FEM using locally refined hp-spaces towards $x = 1$. f is chosen such that the exact solution is $u = \left(\frac{1}{2}\sqrt{2-2x}\right)^3$. Determine the correspondinmg system of linear algeraic equations which you can solve by a direct method. What is the convergence vs. the dimension of the approximation space?

2 Curved elements

Consider the boundary value problem. Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega 2v \quad \forall v \in H_0^1(\Omega)$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, -x^2 - 8 < 4y < x^2 + 8\}$. Discretize this problem by means of the p-Version of the FEM using one element ($p = 2, 3, 4, 5$) and generate the stiffness matrix.

3 Sum Factorization

3.1 Square

Consider the boundary value problem. Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v + uv = \int_\Omega v \quad \forall v \in H_0^1(\Omega)$$

where $\Omega = (0, 1)^2$.

- Discretize this problem by using the p-version of the FEM using 1 element,
- Solve the corresponding system $K\underline{u} = \underline{f}$ by the PCG-method with $C = \mathrm{diag}(K)$ and a relative accuracy of $\varepsilon = 10^{-1-p}$.
- Compare the solution time between an algorithm where the matrix K is computed before the PCG-method using Gaussian quadrature and an algorithm which computes only $K\underline{u}$ within the PCG-method using sum factorization.

3.2 Triangle

Let

$$u_{ij}(x,y) = P_i\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^i P_j^{2i}(y) \quad 0 \le i+j \le p,$$

and

$$M = \left[\int_{-1}^{1}\int_{\frac{y-1}{2}}^{\frac{1-y}{2}} u_{ij}(x,y)u_{kl}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right]_{0\le i+j,k+l\le p}.$$

Write an algorithm in order to compute $M\underline{u}$ with sum factorization in $\mathcal{O}(p^3)$ operations.

4 Assembling the global matrix

Consider the boundary value problem. Find $u \in H^1_\Gamma(\Omega) = \{u \in H^1(\Omega), u(x,1) = u(x,0) = 0\}$ such that

$$\int_\Omega \nabla u \cdot \nabla v + uv = \int_\Omega v \quad \forall v \in H^1_\Gamma(\Omega)$$

where $\Omega = (0,1)^2$. Discretize this problem by using the p-version of the FEM on the mesh of Figure **??**. The element numbers are displayed in red, the edges in blue and the vertices in green color. Compute the stiffness matrix.

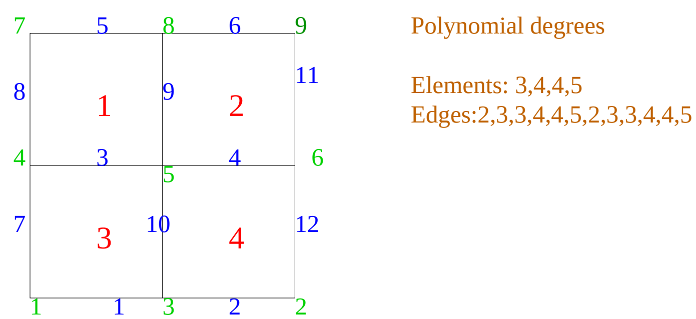

Figure 1: Mesh

A Hints

A.1 Basis functions

If the basis on the elements is not specified, use integrated Legendre polynomials $L_i(x) = \int_{-1}^{x} P_{i-1}(y)\,\mathrm{d}y$, $i \ge 2$ for the higher orders and the hat basis functions $\phi_0(x) = \frac{1-x}{2}$ and $\phi_1(x) = \frac{1+x}{2}$ for the low orders.

The integrated Legendre polynomials satisfy the three-term recurrence

$$L_{i+2}(x) = \frac{2i-1}{i+2}xL_{i+1}(x) - \frac{i-1}{i+2}L_i(x), \quad i \ge 0$$

with initial values $L_0(x) = -1$ and $L_1(x) = x$. This recurrence should be used for the stable evaluation of functional values $L_i(x)$ with a given $x \in [-1,1]$. This you can need for the numerical integration. In addition,

$$L_i(\pm 1) = 0 \quad i \ge 2.$$

Moreover, the relations

$$L_i(x) = \frac{1}{2i-1}(P_i(x) - P_{i-2}(x)), \quad i \geq 3. \tag{A.1}$$

holds, where P_i is the i-th Legendre polynomial.

A.2 Numerical integration

The right hand side in examples 1.3 and 1.4 has to be computed by numerical integration. Since the function f is singular, Gaussian quadrature with the weight $(1-x)^{-1/2}$ should be used (the integration points are the zeros of the Cebyshev polynomials). In the examples 3.1. und 4, the right hand side can be computed explicitely using (**??**) and the orthogonality of the Legendre polynomials. Please keep in mind that the accuracy of the integrals have to be very high.
The stiffness matrix can be computed explicitely in 1 by using (**??**) and the orthogonality of the Legendre polynomials in L_2 and the orthogonality of the integrated Legendre polynomials in H^1. (why?) Finally, you will obtain a banded matrix with maximal bandwidth 4.
For examples 3.1 and 4, it also possible to compute the matrix entries explicitely. Here you should use usual Gaussian quadrature rules as for examples 2 and 3.2.

A.3 Presentation

The exercise is done, if you have programmed it. In addition, you should write about 5 to maximal 10 pages about the exercise including the numerical (or theoretical) results with a short comment about it.