Scientific Computing

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Abstract

1 Literature

1.1 Books about the topic

The algorithmic details presented in part 1 can be found in the monographies [13, 8, 20]. The convergence results of part 2 are from [19].

1.2 Special additional references

- Section 2, Choice of basis functions: [18], [5], [6]
- Section 4, Fast assembling: [17], [16],
- Section 6, Adaptive mesh refinement: [2], [9, 15, 12],
- Section 7, Convergence: [4, 3], regularity results: [10, 7, 11, 14]

2 Mesh grading in 2D

Example of hp-refinement in 2D for corner singularities, domain is the unit square. The grading factor is chosen to be $\sigma = \frac{1}{2}$.

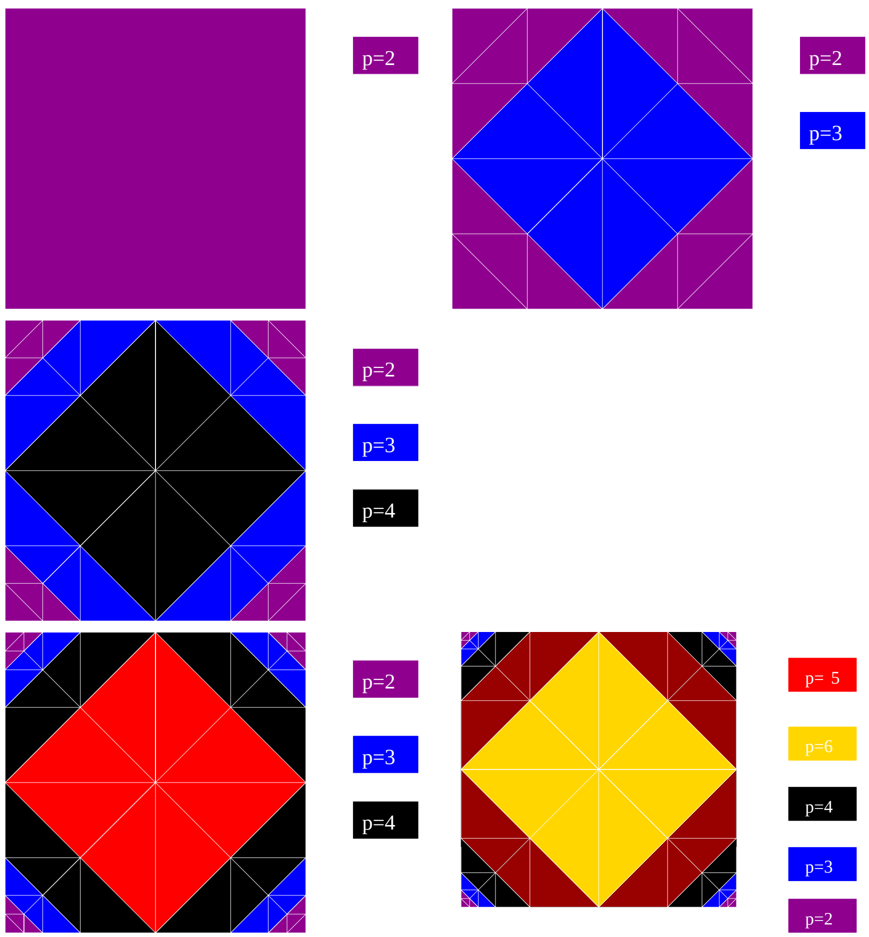

Figure 1: Mesh grading and p- refinement in 4 steps in 2D

A Relations about orthogonal polynomials

In this chapter, we collect some properties of Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!} \frac{1}{(1-x)^\alpha (1+x)^\beta} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right) \tag{A.1}$$

with $\alpha, \beta > -1$. Special cases are

- $\alpha = \beta = 0$, which are called Legendre polynomials $P_n(x)$,
- $\alpha = \beta = \frac{1}{2}$ which are the Chebyshev polynomials of first kind.

It can be proved that

$$\frac{\mathrm{d}}{\mathrm{d}x} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}\,(n+\alpha+\beta+1)\, P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{A.2}$$

In particular

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} P_n(x) = \frac{(n+k)!}{2^k\; n!} P_{n-k}^{(k,k)}(x). \tag{A.3}$$

The Jacobi polynomials are orthogonal to the weighted $L_{2,(1-x)^\alpha(1+x)^\beta}(-1,1)$ scalar product. More precisely,

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\;\mathrm{d}x = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!} \delta_{mn}, \tag{A.4}$$

where $\Gamma(\cdot)$ denotes the Gamma function and δ_{mn} the Kronecker delta. In the special case $m = n = i-k$, $\alpha = \beta = k \in \mathbb{N}$ this relation simplifies to

$$\int_{-1}^{1} (1-x)^k (1+x)^k P_{i-k}^{(k,k)}(x) P_{i-k}^{(k,k)}(x)\;\mathrm{d}x = \frac{2^{2k+1}}{2i+1} \frac{i!i!}{(i+k)!(i-k)!}. \tag{A.5}$$

The Legendre polynomials $P_i(x)$ solves the Legendre differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)^2 \frac{\mathrm{d}u}{\mathrm{d}x}\right) + i(i+1)u(x) = 0. \tag{A.6}$$

The proofs can be found in books of Abramowitz and Stegun, [1], and Tricomi, [21], in the internet or in Lectures of from the RISC concerning orthogonal polynomials.

References

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