Fundamentals: Exercise on Kinematics

Let $\mathbb{DH}$ be the algebra of dual Qquaternions, which is generated over $\mathbb{R}$ by $\mathbf{i}, \mathbf{j}, \mathbf{k}, \epsilon$ with relations

$$\mathbf{i}^2 + 1 = \mathbf{j}^2 + 1 = \mathbf{k}^2 + 1 = \epsilon^2 = 0,$$

$$\mathbf{ij} = \mathbf{k}, \mathbf{ji} = -\mathbf{k}, \mathbf{ki} = \mathbf{j}, \mathbf{ik} = -\mathbf{j}, \mathbf{jk} = \mathbf{i}, \mathbf{kj} = -\mathbf{i},$$

$$\epsilon\mathbf{i} = \mathbf{i}\epsilon, \epsilon\mathbf{j} = \mathbf{j}\epsilon, \epsilon\mathbf{k} = \mathbf{k}\epsilon.$$

The norm $N(h)$ of a dual quaternion h is defined as the product with its conjugate, where

$$\bar{\epsilon} = e, \bar{\mathbf{i}} = -\mathbf{i}, \bar{\mathbf{j}} = -\mathbf{j}, \bar{\mathbf{k}} = -\mathbf{k}.$$

The norm is multiplicative, i.e. $N(ab) = N(a)N(b)$, and it is in general a dual number, i.e. a linear combination of 1 and ϵ.
Let $\mathbb{H}$ be the subspace generated by $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ (the skew field of quaternions). Let V be the subspace generated by $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. Let $\epsilon\mathbb{H}$ be the subspace generated by $(\epsilon, \epsilon\mathbf{i}, \epsilon\mathbf{j}, \epsilon\mathbf{k})$ (a two-sided ideal), which is also the set of all dual quaternions with zero norm. Let $\mathbb{S}$ be the set of all dual quaternions with real norm. Then $\mathbb{S} \setminus \epsilon\mathbb{H}$ together with multiplication is a group.
For any $h = p + \epsilon q$ in $\mathbb{S} \setminus \epsilon\mathbb{H}$ (p and q are supposed to be in $\mathbb{H}$), we define $\alpha(h)$ as the map sending $v \in V$ to $\frac{pv\bar{p}+q\bar{p}}{p\bar{p}}$. Prove that α is group homomorphism from $\mathbb{S} \setminus \epsilon\mathbb{H}$ into the group of Euclidean motions, i.e. the group of all affine linear maps from V to itself preserving distance and orientation. Here the length of a vector in V is defined as the square root of the norm.