I. Warm-up Problems for Todd Quinto's Linz Talks

If you feel so inclined, here are some problems you could think about before my talks. They lead to some calculations I will discuss during the talks. All problems are optional!

1. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function of compact support and assume $\varphi(0) \neq 0$. For $\xi \in \mathbb{R}$, define

$$F(\xi) = \int_0^\infty \varphi(x) e^{-ix\xi} dx.$$

Use integration by parts twice to show that there are constants $c > 0$, $C > 0$, and $K > 0$ such that for $|\xi| \geq K$

$$\frac{c}{|\xi|} \leq |F(\xi)| \leq \frac{C}{|\xi|}.$$

2. Prove that if $f \in L^1(\mathbb{R}^2)$ then for each $x \in \mathbb{R}^2$,

$$\int_{\theta=0}^{2\pi} \int_{t=-\infty}^{\infty} f(x + t\theta^\perp) dt d\theta = f * \frac{2}{|x|}$$

where $\theta^\perp = (-\sin(\theta), \cos(\theta))$ and

$$f * g(x) = \int_{y \in \mathbb{R}^2} f(x - y) g(y) dy.$$

3. Let $\alpha \in (0, \pi)$. Prove that if $f \in L^1(\mathbb{R}^2)$ then for each $x \in \mathbb{R}^2$,

$$\int_{\theta=0}^{\alpha} \int_{t=-\infty}^{\infty} f(x + t\theta^\perp) dt d\theta = f * \frac{T(x)}{|x|}$$

where $T(x)$ is 1 if x is in the cone $\cup\{t\theta^\perp | t \in \mathbb{R}, \theta \in [0, \alpha])$ and 0 otherwise.

II. Definitions and Homework Problems for Linz Talks

If you feel so inclined, below are some problems you could do after the first or second talk. All problems are optional!

First, I give some definitions that I will present in my talks.

Definition 1 *Let* $\theta \in [0, 2\pi]$ *and* $t \in \mathbb{R}$. *Then we define* $\overline{\theta} = (\cos(\theta), \sin(\theta))$ *and* $\theta^\perp = (-\sin(\theta), \cos(\theta))$ *and the line* $L(\theta, p) = \{x \in \mathbb{R}^2 \,|\, x \cdot \overline{\theta} = p\}$.

Let $f \in L^1(\mathbb{R}^2)$. *We define the* Radon transform *of* f *to be*

$$Rf(\theta, p) = \int_{x \in L(\theta, p)} f(x) ds$$

where ds *is arc length measure on the line.*

Let $g \in L^1([0,2\pi]\times\mathbb{R})$. *We define the* dual Radon transform *of* g *to be*

$$R^*g(x) = \int_{\theta=0}^{2\pi} g(\theta, x\cdot\overline{\theta})d\theta$$

where ds *is arc length measure on the line.*

If $\alpha \in (0,\pi)$ *then we define the* limited angle dual Radon transform *of* g *to be*

$$R^*_\alpha g(x) = \int_{\theta=0}^{\alpha} g(\theta, x\cdot\overline{\theta})d\theta$$

Definition 2 *Let* f *be a locally integrable function (or distribution). Let* $(x_0,\xi_0)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}$. *Then,* f is smooth near x_0 in direction ξ_0 *if there exists a cutoff function* φ *at* x_0 *(*$\varphi\in C_c^\infty(\mathbb{R}^n)$ *and* $\varphi(x_0)\neq 0$*) such that the Fourier transform* $\mathcal{F}(\varphi f)(\xi)$ *is rapidly decreasing at* ∞ *(decreasing faster than any power of* $1/|\xi|$*) in* V.

If f *is not smooth near* x_0 *in direction* ξ_0 *then* $(x_0,\xi_0)\in \mathrm{WF}(f)$ *the wavefront set of* f.

4. Use the result of warm-up problem 1 and special cutoff functions to show that if $f(x,y) = \begin{cases} 1 & x\geq 0 \\ 0 & x<0\end{cases}$ then

$$\mathrm{WF}(f) = \{((0,y),(\xi_1,0)\,\big|\,y\in\mathbb{R},\xi_1\neq 0\}.$$

5. Use the result of warm-up problem 1 and special cutoff functions to show that if $f(x,y) = \begin{cases} 1 & x\geq 0 \text{ and } y\geq 0 \\ 0 & \text{otherwise}\end{cases}$ then

$$\mathrm{WF}(f) = \{((0,y),(\xi_1,0)\,\big|\,y>0,\xi_1\neq 0\}\cup\{((x,0),(0,\xi_2)\,\big|\,x>0,\xi_2\neq 0\}$$
$$\cup\{((0,0),(\xi_1,\xi_2)\,\big|\,(\xi_1,\xi_2)\neq 0\}.$$

6. Let $g:\mathbb{R}^n\to\mathbb{R}$ be C^∞. Then, for any function f, $\mathrm{WF}(f)=\mathrm{WF}(f+g)$.

7. Let $x_0\in\mathbb{R}^n$ and let $f=g$ in a neighborhood of x_0. Let $\xi_0\in\mathbb{R}^n$ with $\xi_0\neq 0$. Then, $(x_0,\xi_0)\in\mathrm{WF}(f)$ **iff** $(x_0,\xi_0)\in\mathrm{WF}(g)$.

   $g:\mathbb{R}^n\to\mathbb{R}$ be C^∞. Then, for any function f, $\mathrm{WF}(f)=\mathrm{WF}(f+g)$.

8. Use the result of warm-up problem 2 to show that $R^*Rf = f * \frac{1}{|x|}$.

9. Use the result of warm-up problem 3 to show that $R^*_\alpha Rf = f*\frac{T(x)}{|x|}$ where $T(x)$ is 1 if x is in the cone $\cup\{t\theta^\perp|t\in\mathbb{R},\theta\in[0,\alpha])$.