FUNDAMENTALS
of Numerical Analysis and Symbolic Computation
- Exercises on Ritz' and Trefftz' Methods -

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Exercise 1: Show that the minimization problem

$$\text{Find } u \in V_g : J(u) = \inf_{v \in V_g} J(v) \tag{1}$$

and the variational problem (= variational formulation)

$$\text{Find } u \in V_g : a(u, v) = \langle F, v \rangle \; \forall v \in V_0 \tag{2}$$

are equivalent provided that the bilinear form $a(.,.) : V \times V \to R$ is symmetric and positive, where $J(v) = \frac{1}{2}a(v, v) - \langle f, v \rangle$ is called Ritz' energy functional, and $F \in V_0^*$ is a given linear, bounded functional on V_0.

Exercise 2: Derive the variational formulation of the convection-diffusion-reaction problem (2) from the lectures !

Exercise 3: Show that the Ritz solution $u_h \in V_{gh}$ of the minimization problem

$$J(u_h) = \inf_{v_h \in V_{gh}} J(v_h), \quad \text{with } J(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx, \tag{3}$$

also minimizes the energy functional

$$E(v_h) := \int_\Omega |\nabla (v_h - u)|^2 dx \tag{4}$$

on V_{gh} and vice versa !

Exercise 4: Show that the Ritz solution $u_h = g + \sum_{k=1}^n u_k \varphi_k \in V_{gh}$ of the minimization problem (1) can practically be determined by the solution of the linear system

$$\text{Find } \underline{u}_h = (u_1, \dots, u_n)^T \in R^n : \quad K_h \underline{u}_h = \underline{f}_h \tag{5}$$

with the stiffness matrix $K_h = \big(a(\varphi_k, \varphi_j)\big)_{j,k=1,\dots,n}$ and the load vector $\underline{f}_h = (f_1, \dots, f_n)^T = \big(\langle F, \varphi_1 \rangle, \dots, \langle F, \varphi_n \rangle\big)^T \in R^n$.

Exercise 5: The **Extended Ritz-Method** looks for an approximate solution to the Dirichlet integral minimization problem in the form

$$u_h(x) = g(x) + \sum_{k=1}^{n} u_k p_k(x) \in V_{gh} = g + V_{0h} = g + \text{span}\{p_1, \dots, p_n\}, \tag{6}$$

where $g \in H^1(\Omega)$ is an $H^1(\Omega)$-extension of the given Dirichlet data $g \in H^{1/2}(\Gamma)$, and p_k are subdomain-wise (PDE) harmonic ansatz-functions satisfying the equations

$$-\Delta p_k = 0 \text{ in } \Omega_i,\ \forall i = 1, \dots, p\ (p = 2), \quad p_k = 0 \text{ on } \Gamma,\ p_k = h_k \text{ on } \Gamma_I = \Gamma_S, \tag{7}$$

with k is runing from 1 to n. The given basis functions $h_1, \dots, h_n$ are living on Γ_I and vanishing on $\Gamma_I \cap \Gamma$, i.e.

$$g|_{\Gamma_I \cap \Gamma} + \sum_{k=1}^{n} u_k h_k(x) \approx u. \tag{8}$$

The unknown coefficients $u_1, \dots, u_n$ in ansatz (6) are now chosen such that they solve the minimization problem (3) that is in turn equivalent to the Galerkin equations

$$\int_{\Omega} \nabla u_h(x) \cdot \nabla p_j(x)\, dx = 0 \ \ \forall j = 1, \dots, n. \tag{9}$$

Inserting (6) into (9) and integrating by parts give the extended Ritz equations

$$\sum_{k=1}^{n} u_k \sum_{i=1}^{p} \int_{\Gamma_i} p_k(x) \frac{\partial p_j}{\partial n_i}(x)\, ds = \sum_{i=1}^{p} \int_{\Gamma_i} g(x) \frac{\partial p_j}{\partial n_i}(x)\, ds \ \ \forall j = 1, \dots, n. \tag{10}$$

from which the unknown coefficients $u_1, \dots, u_n$ in the extended Ritz ansatz (6) can be determine. Verify (10) and rewrite the extended Ritz equations (10) for the Trefftz' T-supporter problem given in the lecture, see also [1] !

Exercise 6: The **Extended Trefftz-Method** looks for an approximate solution to the Dirichlet integral minimization problem in the form

$$w_h(x) = p_0(x) + \sum_{k=1}^{n} w_k p_k(x) \notin H^1(\Omega), \tag{11}$$

where the ansatz-functions p_0 and $p_1, \dots, p_n$ satisfy the boundary value problems

$$-\Delta p_0 = 0 \text{ in } \Omega_i,\ \forall i = 1, \dots, p, \quad p_0 = g \text{ on } \Gamma,\ \frac{\partial p_0}{\partial n} = 0 \text{ on } \Gamma_I = \Gamma_S, \tag{12}$$

and

$$-\Delta p_k = 0 \text{ in } \Omega_i,\ \forall i = 1, \dots, p, \quad p_k = 0 \text{ on } \Gamma,\ \frac{\partial p_k}{\partial n} = f_k \text{ on } \Gamma_I = \Gamma_S,\ k = 1, \dots, n \tag{13}$$

with basis funktions $f_k(x)$ defined on Γ_I such that

$$H^{-1/2}(\Gamma_I) \ni \frac{\partial u}{\partial n} \approx \sum_{k=1}^{n} w_k f_k(x), \tag{14}$$

respectively. We mention that the normal n is globally fixed on $\bar{\Omega}_i \cap \bar{\Omega}_j$, i.e. either n_i or n_j. Now we choose the unknown coefficients $w_1, \dots, w_n$ in ansatz (11) such that the broken energy norm functional

$$E_{broken}(w_h) := \sum_{i=1}^{p=2} \int_{\Omega_i} |\nabla(w_h - u)|^2 dx \tag{15}$$

will be minimized. Obviously, w_h must satisfy the equations

$$\frac{\partial E_{broken}}{\partial w_j}(w_h) = 0 \;\; \forall j = 1, \dots, n. \tag{16}$$

Derive the finial system of linear algebraic equations for determining $w_1, \dots, w_n$ for the Trefftz's T-supporter problem, and show that

$$J_{broken}(w_h) := \frac{1}{2} \sum_{i=1}^{p=2} \int_{\Omega_i} |\nabla(w_h)|^2 dx \leq J(u) \leq J(u_h), \tag{17}$$

where u_h is the solution of the extended Ritz equations (10) !

References

[1] E. Trefftz. Ein Gegenstück zum Ritzschen Verfahren. In *Verh. d. 2. Intern. Kongr. f. Techn. Mech.*, 1926. `http://www.unige.ch/ gander/historicalreferences.php`.