Fundamentals of Numerical and Symbolic Computations

Multigrid Methods

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Contents

1 Model problem

Find $u\colon [0,1] \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} -u''(x) &= f(x) \quad \text{for all } x \in (0,1), \\ u(0 &= u(1) = 0 \end{aligned}$$

for given $f\colon (0,1) \longrightarrow \mathbb{R}$.

Discretization: Let $n \in \mathbb{N}$. We choose nodes x_i with

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1.$$

By these nodes the interval $[0,1]$ is divided into n subintervals. For simplicity we consider equidistant subdivisions only:

$$x_i = i \cdot h \quad \text{with} \quad h = \frac{1}{n}.$$

Discretization by a finite difference method:

$$u''(x) \approx \frac{1}{h^2}\big(u(x-h) - 2u(x) + u(x+h)\big).$$

This leads to a system of $n-1$ equations

$$\frac{1}{h^2}\big(u_{i-1} - 2u_i + u_{i+1}\big) = f(x_i) \quad \text{for all } i = 1, 2, \ldots, n-1$$

for the unknowns u_i for $i = 1, \ldots, n-1$. Here u_i denotes the approximation to $u(x_i)$. Observe that it is natural to set $u_0 = u_n = 0$ because of the boundary conditions. In matrix-vector notation:

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}}_{K_h} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix}}_{\underline{u}_h} = \underbrace{\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{\underline{f}_h}$$

Typical properties of K_h: large, sparse and, in the case considered here, symmetric.

2 ω-Jacobi method

Let $\omega > 0$. The ω-Jacobi method is a simple iterative method for solving the linear system

$$K_h \underline{u}_h = \underline{f}_h.$$

It reads:

$$\underline{u}_h^{(j+1)} = \underline{u}_h^{(j)} + \omega\, D_h^{-1} \left(\underline{f}_h - K_h \underline{u}_h^{(j)}\right),$$

with $D_h = \operatorname{diag} K_h = \frac{2}{h^2} I$. Hence, the ω-Jacobi method can be written as:

$$\underline{u}_h^{(j+1)} = \underline{u}_h^{(j)} + \frac{\omega h^2}{2} \left(\underline{f}_h - K_h \underline{u}_h^{(j)}\right).$$

2.1 Convergence analysis

We have

$$\underline{u}_h^{(j+1)} = \underbrace{\left[I - \frac{\omega h^2}{2} K\right]}_{S_h} \underline{u}_h^{(j)} + \frac{\omega h^2}{2} \underline{f}_h = \mathcal{S}_h\left(\underline{u}_h^{(j)}, \underline{f}_h\right),$$

Let $\underline{u}_h^*$ be the exact solution:

$$K_h \underline{u}_h^* = \underline{f}_h$$

and let $\underline{z}_j^{(j)}$ be the error of the k-th iterate:

$$\underline{z}_j^{(j)} = \underline{u}_h^* - \underline{u}_h^{(j)}.$$

Then we have

$$\begin{aligned}\underline{z}_j^{(j+1)} = \underline{u}_h^* - \underline{u}_h^{(j+1)} &= \underline{u}_h^* - \underline{u}_h^{(j)} - \frac{\omega h^2}{2}\left(\underline{f}_h - K_h \underline{u}_h^{(j)}\right) \\ &= \underline{u}_h^* - \underline{u}_h^{(j)} - \frac{\omega h^2}{2} K_h \left(\underline{u}_h^* - \underline{u}_h^{(j)}\right) = \left[I - \frac{\omega h^2}{2} K_h\right] \underline{z}_j^{(j)}\end{aligned}$$

and, therefore,

$$\underline{z}_j^{(j+1)} = S_h \underline{z}_h^{(j)}.$$

2.2 Discrete Fourier transform

For, $k = 1, \ldots, n-1$, we introduce the vectors

$$\underline{\varphi}_{h,k} = \left(\sqrt{2h}\,\sin(k\pi x_i)\right)_{i=1,\ldots,n-1} \in \mathbb{R}^{n-1}.$$

It is easy to see that

$$\left\langle \underline{\varphi}_{h,k}, \underline{\varphi}_{h,l} \right\rangle = \delta_{k,l},$$

where $\langle .,. \rangle$ denotes the Euclidean inner product and δ_{kl} denotes Kronecker's delta.

So $\{\underline{\varphi}_{h,k} \colon k = 1, \ldots, n-1\}$ is a orthonormal basis of $\mathbb{R}^{n-1}$.

Moreover, $\underline{\varphi}_{h,k}$ is an eigenvector of K_h:

$$K_h \underline{\varphi}_{h,k} = \lambda_{h,k}\, \underline{\varphi}_{h,k} \quad \text{with} \quad \lambda_{h,k} = \frac{4}{h^2} \sin^2\left(\frac{1}{2} k\pi h\right).$$

We have

$$0 < \lambda_{h,1} < \ldots < \lambda_{h,k} < \ldots < \lambda_{h,n-1}$$

For the extreme eigenvalues we obtain for $h \ll 1$:

$$\lambda_{h,1} = \frac{4}{h^2} \sin^2\left(\frac{1}{2}\pi h\right) \approx \frac{4}{h^2}\, \frac{1}{4} \pi^2 h^2 = \pi^2$$

and

$$\lambda_{h,n-1} = \frac{4}{h^2}\,\sin^2\left(\frac{n-1}{2}\pi h\right) = \frac{4}{h^2}\,\sin^2\left(\frac{\pi}{2}-\frac{1}{2}\pi h\right) = \frac{4}{h^2}\,\cos^2\left(\frac{1}{2}\pi h\right) \approx \frac{4}{h^2}$$

Since $\{\underline{\varphi}_{h,k} \colon k = 1,\ldots,n-1\}$ is a basis of $\mathbb{R}^{n-1}$, a vector $\underline{z}_h = (z_i)_{i=1,\ldots,n-1} \in \mathbb{R}^{n-1}$ can be uniquely written in the following form:

$$\underline{z}_h = \sum_{k=1}^{n-1} \hat{z}_k\, \underline{\varphi}_{h,k} \tag{1}$$

with coefficients $\hat{z}_k \in \mathbb{R}$ that define a vector $\underline{\hat{z}}_h = (\hat{z}_k)_{k=1,\ldots,n-1} \in \mathbb{R}^{n-1}$.

Interpretation of (1): Decomposition of the vector $\underline{z}_h$ in modes of different frequency. Small values of k (close to 1) correspond to low-frequency modes, large values of k (close to $n-1$) correspond to high-frequency modes.

One can show that the Fourier transform $\underline{z}_h \mapsto \underline{\hat{z}}_h$ is isometric, i.e.:

$$\|\underline{z}_h\| = \|\underline{\hat{z}}_h\|,$$

where $\|.\|$ denotes the Euclidean norm.

2.3 Fourier analysis

We use the discrete Fourier transform to represent the errors in the following form:

$$\underline{z}_h^{(j+1)} = \sum_{k=1}^{n-1} \hat{z}_k^{(j+1)}\, \underline{\varphi}_{h,k} \quad \text{and} \quad \underline{z}_h^{(j)} = \sum_{k=1}^{n-1} \hat{z}_k^{(j)}\, \underline{\varphi}_{h,k}$$

Then

$$\begin{aligned}\sum_{k=1}^{n-1} \hat{z}_k^{(j+1)}\, \underline{\varphi}_{h,k} = \underline{z}_h^{(j+1)} &= \left[I - \frac{\omega h^2}{2} K_h\right] \underline{z}_h^{(j)} = \left[I - \frac{\omega h^2}{2} K_h\right] \sum_{k=1}^{n-1} \hat{z}_k^{(j)}\, \underline{\varphi}_{h,k} \\ &= \sum_{k=1}^{n-1} \left[I - \frac{\omega h^2}{2} K_h\right] \hat{z}_k^{(j)}\, \underline{\varphi}_{h,k} = \sum_{k=1}^{n-1} \left[I - \frac{\omega h^2}{2} \lambda_{h,k}\right] \hat{z}_k^{(j)}\, \underline{\varphi}_{h,k}\end{aligned}$$

This implies

$$\hat{z}_k^{(j+1)} = \left[I - \frac{\omega h^2}{2} \lambda_{h,k}\right] \hat{z}_k^{(j)}$$

and

$$\begin{aligned}\left\|\hat{z}_k^{(j+1)}\right\|^2 &= \sum_{k=1}^{n-1} \left(\hat{z}_k^{(j+1)}\right)^2 = \sum_{k=1}^{n-1} \left[I - \frac{\omega h^2}{2} \lambda_{h,k}\right]^2 \left(\hat{z}_k^{(j)}\right)^2 \\ &\le \max_{k=1,\ldots,n-1} \left[I - \frac{\omega h^2}{2} \lambda_{h,k}\right]^2 \sum_{k=1}^{n-1} \left(\hat{z}_k^{(j)}\right)^2 \left\|\hat{z}_k^{(j)}\right\|^2 = \max_{k=1,\ldots,n-1} \left[1 - \frac{\omega h^2}{2} \lambda_{h,k}\right]^2 \left\|\hat{z}_k^{(j)}\right\|^2\end{aligned}$$

It follows that

$$\left\| z_k^{(j+1)} \right\| = \left\| \hat{z}_k^{(j+1)} \right\| \le \rho(h,\omega) \left\| \hat{z}_k^{(j)} \right\| = \left\| z_k^{(j)} \right\|$$

with

$$\rho(h,\omega) = \max_{k=1,\dots,n-1} \left| 1 - \frac{\omega h^2}{2} \lambda_{h,k} \right|$$

One can show that $\rho(h,\omega) = \rho(S_h) = \|S_h\|$, where $\rho(S_h)$ and $\|S_h\|$ denote the spectral radius and the spectral norm of S_h, respectively.

Properties of $\rho(h,\omega)$:

$$\rho(h,\omega) < 1 \Longleftrightarrow 0 < \omega < \frac{4}{h^2 \lambda_{h,n-1}} = \frac{1}{\sin^2\left(\frac{1}{2}(n-1)\pi h\right)} = \frac{1}{\cos^2\left(\frac{1}{2}\pi h\right)}$$

and

$$\rho(h,\omega_{\text{opt}}) = \min_{\omega} \rho(h,\omega)$$

for

$$\omega_{\text{opt}} = \frac{4}{h^2\left(\lambda_{h,1} + \lambda_{h,n-1}\right)} = \frac{1}{\sin^2\left(\frac{1}{2}\pi h\right) + \cos^2\left(\frac{1}{2}\pi h\right)} = 1$$

$$\rho_{\text{opt}} = \rho(h,\omega_{\text{opt}}) = 1 - 2\,\sin^2\left(\frac{1}{2}\pi h\right) \approx 1$$

So, the method has a very slow convergence rate for $h \ll 1$.

However, different modes of the error (corresponding to different values of k) show different convergence behavior, described by the reduction factor

$$\left| 1 - \frac{\omega h^2}{2} \lambda_{h,k} \right| = \left| 1 - 2\omega \sin^2\left(\frac{1}{2} k\pi h\right) \right| = \left| 1 - \omega\left(1 - \cos\left(k\pi h\right)\right) \right|$$

Case $\omega = 1$: Reduction factor

$$\left|\cos\left(k\pi h\right)\right|$$

Therefore,

- slow convergence for k close to 1 or $n-1$
- fast convergence for modes k close to $\frac{n}{2}$

Case $\omega = \frac{1}{2}$: Reduction factor

$$\left| \frac{1}{2} + \frac{1}{2} \cos\left(k\pi h\right) \right| = \cos^2\left(\frac{1}{2} k\pi h\right)$$

Therefore,

- slow convergence for the high-frequency modes

- fast convergence for the low-frequency modes. Observe that

$$\cos^2\left(\frac{1}{2}k\pi h\right) \le \frac{1}{2} \quad \text{for} \quad k \ge \frac{n}{2}.$$

So, after a few iterates, the error may not be small but high-frequency modes are strongly reduced: the error is smooth.

Smoothing rate:

$$\mu(h,\omega) = \max_{k\ge\frac{n}{2}} \left|1 - \frac{\omega h^2}{2}\lambda_{h,k}\right|$$

In summary, we have for $\omega = \frac{1}{2}$:

$$\hat{z}_k^{(j+1)} = c_{h,k}^2\, \hat{z}_k^{(j)} \quad \text{with} \quad c_{h,k} = \cos\left(\frac{1}{2}k\pi h\right) \quad \text{for all} \quad k = 1,\ldots,n-1$$

resulting in

$$\rho\left(h,\frac{1}{2}\right) \approx 1 \quad \text{and} \quad \mu\left(h,\frac{1}{2}\right) = \frac{1}{2}.$$

If k is replaced by $n-k$, it follows

$$\hat{z}_{n-k}^{(j+1)} = c_{h,n-k}^2\, \hat{z}_{n-k}^{(j)}$$

with

$$c_{h,n-k} = \cos\left(\frac{1}{2}(n-k)\pi h\right) = \sin\left(\frac{1}{2}k\pi h\right)$$

Therefore,

$$\hat{z}_{n-k}^{(j+1)} = s_{h,k}^2\, \hat{z}_{n-k}^{(j)} \quad \text{with} \quad s_{h,k} = \sin\left(\frac{1}{2}k\pi h\right) \quad \text{for all} \quad k = 1,\ldots,n-1$$

In particular, it follows for n even that

$$\hat{z}_{\frac{n}{2}}^{(j+1)} = \frac{1}{2}\,\hat{z}_{\frac{n}{2}}^{(j)},$$

since

$$c_{h,\frac{n}{2}}^2 = \cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}.$$

3 A two-grid method

3.1 The algorithm

Let $\underline{u}_h^{(0)}$ be an initial guess of the exact solution $\underline{u}_h^*$. We proceed as follows to compute the next iterate $\underline{u}_h^{(1)}$ of the two-grid method:

- **Smoothing step**

  First a few , say ν steps, of the ω-Jacobi method are performed:

  $$\underline{u}_h^{(0,j+1)} = \mathcal{S}_h\left(\underline{u}_h^{(0,j)}, \underline{f}_h\right) \quad \text{for} \quad j = 0, \dots, \nu - 1 \quad \text{with} \quad \underline{u}_h^{(0,0)} = \underline{u}_h^{(0)}.$$

  Then the ideal correction $\underline{z}_h$, given by

  $$\underline{u}_h^* = \underline{u}_h^{(0,\nu)} + \underline{z}_h,$$

  is the solution of the residual equation

  $$K_h \underline{z}_h = \underline{f}_h - K_h \underline{u}_h^{(0,\nu)} \equiv \underline{r}_h \tag{2}$$

  In the following second step of the method an approximation of this ideal correction is computed.

- **Coarse grid correction**

  We have seen that the error is smooth after a few step of the ω-Jacobi method. Therefore, $\underline{r}_h$ can be represented on a coarser mesh very accurately.

  - **Restriction**

    We assume $n = 2N$ and consider a coarse grid with the following nodes:

    $$X_I = I \cdot H \quad \text{with} \quad H = \frac{1}{N} \quad \text{for} \quad I = 0, 1, \dots, N.$$

    Obviously, we have $X_I = x_i$ for $i = 2I$.
    We define a coarse grid representation $\underline{r}_H$ of $\underline{r}_h$ by the following averaging

    $$r_{H,I} = \frac{1}{4} r_{h,i-1} + \frac{1}{2} r_{h,i} + \frac{1}{4} r_{h,i+1} \quad \text{with} \quad i = 2I$$

    or in matrix-vector notation

    $$\underline{r}_H = I_h^H \underline{r}_h$$

    with an appropriate $(N-1)$-by-$(n-1)$ matrix I_h^H.

  - **Coarse grid correction equation**

    Instead of the residual equation (2) on the original (fine) grid we solve the corresponding equation on the coarse grid:

    $$K_H \underline{w}_H = \underline{r}_H \tag{3}$$

    with the $(N-1)$-by-$(N-1)$ matrix

    $$K_H = \frac{1}{H^2} \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

    The correction $\underline{w}_H$ is computed by solving (3) exactly.

- **Prolongation**
  The correction $\underline{w}_H$ has to be extended to a vector $\underline{w}_h$ on the fine grid. This is done by interpolation:

$$w_{h,i} = w_{H,I} \quad \text{and} \quad w_{h,i+1} = \frac{1}{2}\left(w_{H,I} + w_{H,I+1}\right) \quad \text{for} \quad i = 2I$$

  or, in matrix-vector notation

$$\underline{w}_h = I_H^h \underline{w}_H.$$

  with an appropriate $(n-1)$-by-$(N-1)$ matrix I_H^h.

We use $\underline{w}_h$ instead of $\underline{z}_h$ for defining the next iterate of one step of the two-grid method:

$$\underline{u}_h^{(1)} = \underline{u}_h^{(0,\nu)} + \underline{w}_h$$

In summary, we obtain

$$\begin{aligned}\underline{u}_h^{(1)} &= \underline{u}_h^{(0,\nu)} + \underline{w}_h = \underline{u}_h^{(0,\nu)} + I_H^h \underline{w}_H = \underline{u}_h^{(0,\nu)} + I_H^h K_H^{-1} \underline{r}_H = \underline{u}_h^{(0,\nu)} + I_H^h K_H^{-1} I_h^H \underline{r}_h \\ &= \underline{u}_h^{(0,\nu)} + I_H^h K_H^{-1} I_h^H \left(\underline{f}_h - K_h \underline{u}_h^{(0,\nu)}\right)\end{aligned}$$

This leads to the following relation for the associated errors:

$$\begin{aligned}\underline{z}_h^{(1)} &= \underline{u}_h^* - \underline{u}_h^{(1)} = \underline{u}_h^* - \underline{u}_h^{(0)} - I_H^h K_H^{-1} I_h^H \left(\underline{f}_h - K_h \underline{u}_h^{(0,\nu)}\right) \\ &= \underline{u}_h^* - \underline{u}_h^{(0)} - I_H^h K_H^{-1} I_h^H K_h \left(\underline{u}_h^* - \underline{u}_h^{(0,\nu)}\right) = \left[I - I_H^h K_H^{-1} I_h^H K_h\right] \underline{z}_h^{(0,\nu)} \\ &= \underbrace{\left[I - I_H^h K_H^{-1} I_h^H K_h\right] S_h^\nu}_{M_h^{TGM}} \underline{z}_h^{(0)}\end{aligned}$$

3.2 Fourier analysis

For

$$\underline{z}_h^{(0)} = \sum_{k=1}^{n-1} \hat{z}_{h,k}^{(0)} \underline{\varphi}_{h,k}, \quad \underline{z}_h^{(0,\nu)} = \sum_{k=1}^{n-1} \hat{z}_{h,k}^{(0,\nu)} \underline{\varphi}_{h,k} \quad \text{and} \quad \underline{z}_h^{(1)} = \sum_{k=1}^{n-1} \hat{z}_{h,k}^{(1)} \underline{\varphi}_{h,k}$$

it follows that

$$\begin{aligned}\hat{z}_{h,k}^{(1)} &= s_{h,k}^2 \hat{z}_{h,k}^{(0,\nu)} + c_{h,k}^2 \hat{z}_{h,n-k}^{(0,\nu)} \quad \text{for all} \quad k = 1, \ldots, N-1 \\ \hat{z}_{h,N}^{(1)} &= \hat{z}_{h,N}^{(0,\nu)} \\ \hat{z}_{h,n-k}^{(1)} &= s_{h,k}^2 \hat{z}_{h,k}^{(0,\nu)} + c_{h,k}^2 \hat{z}_{h,n-k}^{(0,\nu)} \quad \text{for all} \quad k = 1, \ldots, N-1\end{aligned} \tag{4}$$

Therefore,

- fast convergence for high-frequency modes
- slow convergence or stagnation for the other modes.

From the Fourier analysis of the ω-Jacobi method we know that

$$\begin{aligned}
\hat{z}_{h,k}^{(0,\nu)} &= c_{h,k}^{2\nu}\,\hat{z}_{h,n-k}^{(0)} \quad \text{for all} \quad k = 1,\ldots,N-1 \\
\hat{z}_{h,N}^{(0,\nu)} &= \frac{1}{2^\nu}\,\hat{z}_{h,N}^{(0)} \\
\hat{z}_{h,n-k}^{(0,\nu)} &= s_{h,k}^{2\nu}\,\hat{z}_{h,n-k}^{(0)} \quad \text{for all} \quad k = 1,\ldots,N-1,
\end{aligned}$$

since $c_{h,n-k} = s_{h,k}$. Therefore,

$$\begin{aligned}
\hat{z}_{h,k}^{(1)} &= s_{h,k}^2\,c_{h,k}^{2\nu}\,\hat{z}_{h,n-k}^{(0)} + c_{h,k}^2\,s_{h,k}^{2\nu}\,\hat{z}_{h,n-k}^{(0)} \quad \text{for all} \quad k = 1,\ldots,N-1 \\
\hat{z}_{h,N}^{(1)} &= \frac{1}{2^\nu}\,\hat{z}_{h,N}^{(0)} \\
\hat{z}_{h,n-k}^{(1)} &= s_{h,k}^2\,c_{h,k}^{2\nu}\,\hat{z}_{h,n-k}^{(0)} + c_{h,k}^2\,s_{h,k}^{2\nu}\,\hat{z}_{h,n-k}^{(0)} \quad \text{for all} \quad k = 1,\ldots,N-1
\end{aligned}$$

Then, for $k = 1,\ldots,N-1$,

$$\begin{aligned}
\begin{bmatrix} \hat{z}_{h,k}^{(1)} \\ \hat{z}_{h,n-k}^{(1)} \end{bmatrix} &= \begin{bmatrix} s_{h,k}^2\,c_{h,k}^{2\nu} & c_{h,k}^2\,s_{h,k}^{2\nu} \\ s_{h,k}^2\,c_{h,k}^{2\nu} & c_{h,k}^2\,s_{h,k}^{2\nu} \end{bmatrix} \begin{bmatrix} \hat{z}_{h,k}^{(1)} \\ \hat{z}_{h,n-k}^{(1)} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s_{h,k}^2\,c_{h,k}^{2\nu} & 0 \\ 0 & c_{h,k}^2\,s_{h,k}^{2\nu} \end{bmatrix} \begin{bmatrix} \hat{z}_{h,k}^{(0)} \\ \hat{z}_{h,n-k}^{(0)} \end{bmatrix}
\end{aligned}$$

Now

$$s_{h,k}^2\,c_{h,k}^{2\nu} = \phi_\nu(c_{h,k}^2) \quad \text{and} \quad c_{h,k}^2\,s_{h,k}^{2\nu} = \phi_\nu(s_{h,k}^2) \quad \text{with} \quad \phi_\nu(x) = (1-x)x^\nu$$

By elementary calculations one can show that

$$\max_{x\in[0,1]} \phi_\nu(x) = \frac{\nu^\nu}{(\nu+1)^{\nu+1}} \leq \frac{1}{e\,\nu}$$

Therefore,

$$\left\| \begin{bmatrix} \hat{z}_{h,k}^{(1)} \\ \hat{z}_{h,n-k}^{(1)} \end{bmatrix} \right\| \leq \underbrace{\left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\|}_{=2} \underbrace{\left\| \begin{bmatrix} s_{h,k}^2\,c_{h,k}^{2\nu} & 0 \\ 0 & c_{h,k}^2\,s_{h,k}^{2\nu} \end{bmatrix} \right\|}_{\leq \frac{\nu^\nu}{(\nu+1)^{\nu+1}}} \left\| \begin{bmatrix} \hat{z}_{h,k}^{(0)} \\ \hat{z}_{h,n-k}^{(0)} \end{bmatrix} \right\|$$

This implies

$$\begin{aligned}
\left\| \underline{z}_h^{(1)} \right\|^2 &= \left\| \hat{\underline{z}}_h^{(1)} \right\|^2 = \sum_{k=1}^{N-1} \left\| \begin{bmatrix} \hat{z}_{h,k}^{(1)} \\ \hat{z}_{h,n-k}^{(1)} \end{bmatrix} \right\|^2 + |\hat{z}_{h,N}^{(1)}|^2 \\
&\leq \left[\frac{2\nu^\nu}{(\nu+1)^{\nu+1}} \right]^2 \sum_{k=1}^{N-1} \left\| \begin{bmatrix} \hat{z}_{h,k}^{(0)} \\ \hat{z}_{h,n-k}^{(0)} \end{bmatrix} \right\|^2 + \left[\frac{1}{2^\nu} \right]^2 |\hat{z}_{h,N}^{(0)}|^2 \\
&\leq \left[\frac{2\nu^\nu}{(\nu+1)^{\nu+1}} \right]^2 \left\{ \sum_{k=1}^{N-1} \left\| \begin{bmatrix} \hat{z}_{h,k}^{(0)} \\ \hat{z}_{h,n-k}^{(0)} \end{bmatrix} \right\|^2 + |\hat{z}_{h,N}^{(0)}|^2 \right\} = \left[\frac{2\nu^\nu}{(\nu+1)^{\nu+1}} \right]^2 \left\| \hat{\underline{z}}_h^{(0)} \right\|^2 \\
&= \left[\frac{2\nu^\nu}{(\nu+1)^{\nu+1}} \right]^2 \left\| \underline{z}_h^{(0)} \right\|^2
\end{aligned}$$

Hence

$$\left\| \underline{z}_h^{(1)} \right\| \leq \frac{2\nu^\nu}{(\nu+1)^{\nu+1}} \left\| \underline{z}_h^{(0)} \right\|$$

This shows that the two-grid method converges with a rate that is uniformly smaller than 1 (independent of h). The convergence rate decreases if ν increases:

$$\frac{2\nu^\nu}{(\nu+1)^{\nu+1}} \leq \frac{2}{e\,\nu}.$$

4 Multigrid methods

Recursive application of the two-grid idea.

- V-cycle
- W-cycle

5 Exercises

Let $N \in \mathbb{N}$ and set

$$n = 2N,\ h = \frac{1}{n},\ x_i = i\,h, \quad H = \frac{1}{N} = 2h,\ X_I = IH = 2Ih = x_{2I} = x_i \text{ for } i = 2I.$$

The following abbreviations are introduced:

$$s_{h,k} = \sin\left(\frac{1}{2}k\pi h\right), \quad c_{h,k} = \cos\left(\frac{1}{2}k\pi h\right) \quad \text{for} \quad k = 1, \ldots, n-1$$

and

$$s_{H,k} = \sin\left(\frac{1}{2}k\pi H\right), \quad c_{H,k} = \cos\left(\frac{1}{2}k\pi H\right) \quad \text{for} \quad k = 1, \ldots, N-1.$$

For $k = 1, \dots, n-1$ the vectors $\underline{\varphi}_{n,k}$, given by

$$\underline{\varphi}_{h,k} = \left(\sqrt{2h}\, \sin(k\pi x_i)\right)_{i=1,\dots,n-1} \in \mathbb{R}^{n-1},$$

build an orthonormal basis in $\mathbb{R}^{n-1}$. For $k = 1, \dots, N-1$ the vectors $\underline{\varphi}_{H,k}$, given by

$$\underline{\varphi}_{H,k} = \left(\sqrt{2H}\, \sin(k\pi X_i)\right)_{i=1,\dots,N-1} \in \mathbb{R}^{N-1},$$

build an orthonormal basis in $\mathbb{R}^{N-1}$.

- Show that

  $$K_h \underline{\varphi}_{h,k} = \lambda_{h,k}\, \underline{\varphi}_{h,k} \quad \text{with} \quad \lambda_{h,k} = \frac{4}{h^2}\, s_{h,k}^2 \quad \text{for} \quad k = 1, \dots, n-1$$

  and

  $$K_H \underline{\varphi}_{H,k} = \lambda_{H,k}\, \underline{\varphi}_{H,k} \quad \text{with} \quad \lambda_{H,k} = \frac{4}{H^2}\, s_{H,k}^2 \quad \text{for} \quad k = 1, \dots, N-1.$$

  Hint: $\sin(x-y) + \sin(x+y) = 2 \sin x \cos y$.

- Show:

  $$\text{If} \quad \underline{z}_h = \sum_{k=1}^{n-1} \alpha_k\, \underline{\varphi}_{h,k}, \quad \text{then} \quad K_h \underline{z}_h = \sum_{k=1}^{n-1} \alpha'_k\, \underline{\varphi}_{h,k} \quad \text{with} \quad \alpha'_k = \lambda_{h,k}\, \alpha_k$$

  for $k = 1, \dots, n-1$.

For the fine-grid vector $\underline{r}_h = (r_{h,i})_{i=1,\dots,n-1}$ its restriction $\underline{r}_H = (r_{H,I})_{I=1,\dots,N-1}$ to the coarse grid is given by

$$r_{H,I} = \frac{1}{4}\, r_{h,i-1} + \frac{1}{2}\, r_{h,i} + \frac{1}{4}\, r_{h,i+1} \quad \text{with} \quad i = 2I,$$

in short:

$$\underline{r}_H = I_h^H \underline{r}_h.$$

- Show that

  $$\begin{aligned} I_h^H \underline{\varphi}_{h,k} &= \frac{1}{\sqrt{2}}\, c_{h,k}^2\, \underline{\varphi}_{H,k} \quad \text{for all } k = 1, \dots, N-1, \\ I_h^H \underline{\varphi}_{h,N} &= 0, \\ I_h^H \underline{\varphi}_{h,n-k} &= -\frac{1}{\sqrt{2}}\, s_{h,k}^2\, \underline{\varphi}_{H,k} \quad \text{for all } k = 1, \dots, N-1 \end{aligned}$$

  Hint for the last identity: Show first: the i-th component of $\underline{\varphi}_{h,n-k}$ is equal to $-(-1)^i$ times the i-th component of $\underline{\varphi}_{h,k}$.

- Show:

$$\text{If} \quad \underline{r}_h = \sum_{k=1}^{n-1} \beta_k \, \underline{\varphi}_{h,k}, \quad \text{then} \quad I_h^H \underline{r}_h = \sum_{k=1}^{N-1} \beta'_k \, \varphi_{H,k} \quad \text{with} \quad \beta'_k = \frac{1}{\sqrt{2}} \left[c_{h,k}^2 \, \beta_k - s_{h,k}^2 \, \beta_{n-k} \right].$$

The coarse grid correction equation reads.

$$K_H \underline{w}_H = \underline{r}_H$$

- Show:

$$\text{If} \quad \underline{r}_H = \sum_{k=1}^{N-1} \gamma_k \, \underline{\varphi}_{H,k}, \quad \text{then} \quad \underline{w}_H = \sum_{k=1}^{N-1} \gamma'_k \, \underline{\varphi}_{H,k} \quad \text{with} \quad \gamma'_k = \frac{1}{\lambda_{H,k}} \, \gamma_k.$$

The prolongation of the coarse-grid vector $\underline{w}_H = (w_{H,I})_{I=1,\dots,N-1}$ to a fine-grid vector $\underline{w}_h = (w_{h,i})_{i=1,\dots,n-1}$ is given by

$$\begin{aligned} w_{h,i} &= w_{H,I} \quad \text{for all } I = 1, \dots, N-1 \\ w_{h,i+1} &= \frac{1}{2} \left[w_{H,I} + w_{H,I+1} \right] \quad \text{for all } I = 0, \dots, N \end{aligned}$$

with $i = 2I$, in short:

$$\underline{w}_h = I_H^h \underline{w}_H.$$

- Show:

$$I_H^h \underline{\varphi}_{H,k} = \sqrt{2} \left[c_{h,k}^2 \, \underline{\varphi}_{h,k} - s_{h,k}^2 \, \underline{\varphi}_{h,n-k} \right] \quad \text{for all } k = 1, \dots, N-1.$$

- Show:

$$\text{If} \quad \underline{w}_H = \sum_{k=1}^{N-1} \delta_k \, \underline{\varphi}_{H,k}, \quad \text{then} \quad w_h = I_H^h \underline{w}_H = \sum_{k=1}^{n-1} \delta'_k \, \underline{\varphi}_{h,k}$$

  with

$$\begin{aligned} \delta'_k &= \sqrt{2} \, c_{h,k}^2 \, \delta_k \quad \text{for all } k = 1, \dots, N-1, \\ \delta'_N &= 0, \\ \delta'_{n-k} &= -\sqrt{2} \, s_{h,k}^2 \, \delta_k \quad \text{for all } k = 1, \dots, N-1. \end{aligned}$$

- Use these results for the action of K_h, I_h^H, K_H^{-1}, and I_H^h, and show (4).

References

[1] W. Hackbusch, *Multi-Grid Methods and Applications*, Berlin: Springer-Verlag, 1985.

[2] W. Hackbusch, *Iterative Solutions of Large Sparse Systems of Equations*, New York: Springer-Verlag, 1994.