The Concrete Tetrahedron

Manuel Kauers · RISC

Introduction

Recall: Quicksort

Recall: Quicksort

a_1	a_2	a_3										a_n

Recall: Quicksort

Recall: Quicksort

Recall: Quicksort

a_1	a_2	a_3										a_n

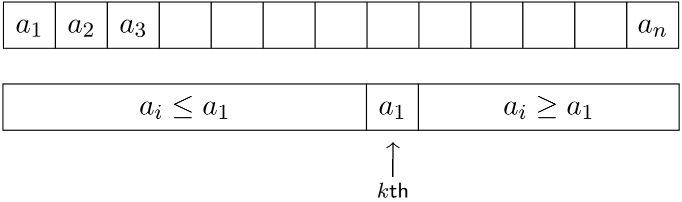
$a_i \leq a_1$		$a_i \geq a_1$

Recall: Quicksort

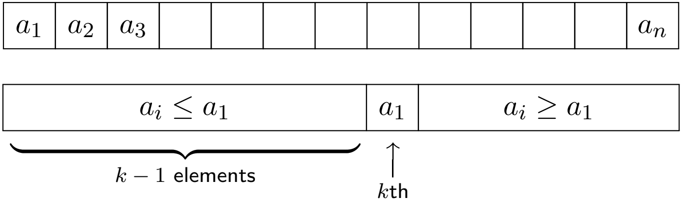
a_1	a_2	a_3										a_n

$a_i \leq a_1$	a_1	$a_i \geq a_1$

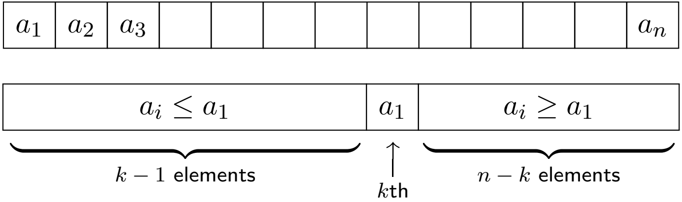
Recall: Quicksort



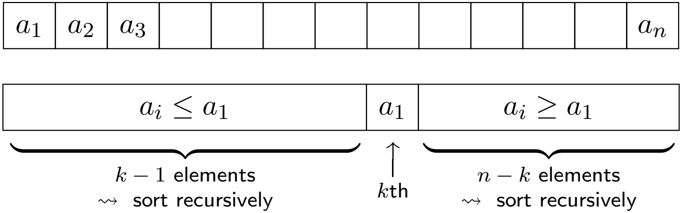
Recall: Quicksort



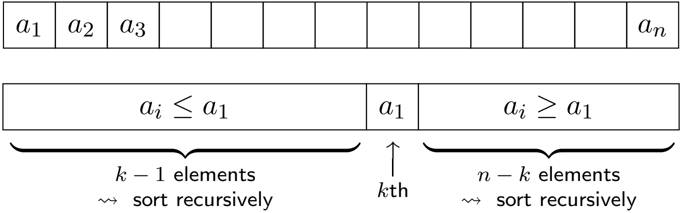
Recall: Quicksort



Recall: Quicksort



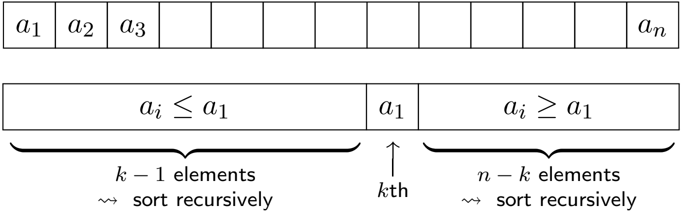
Recall: Quicksort



If c_n is the *average number* of comparisons, then

$$c_n =$$

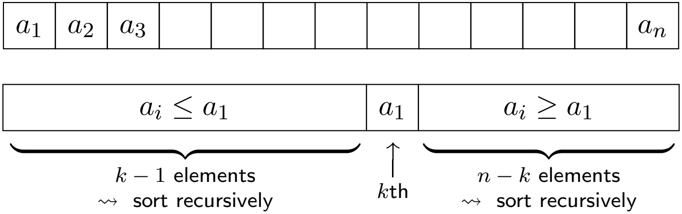
Recall: Quicksort



If c_n is the *average number* of comparisons, then

$$c_n = (n-1) +$$

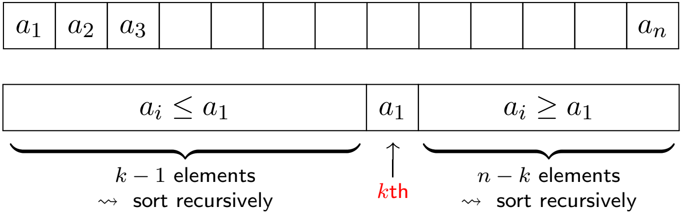
Recall: Quicksort



If c_n is the *average number* of comparisons, then

$$c_n = (n-1) + \frac{1}{n} \sum_{k=1}^{n} \left(c_{k-1} + c_{n-k} \right)$$

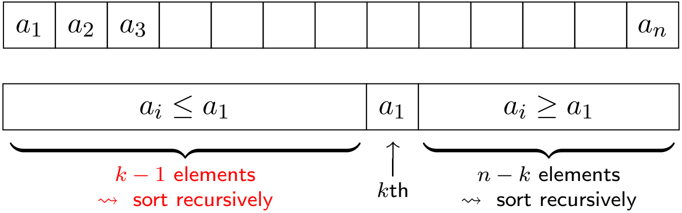
Recall: Quicksort



If c_n is the *average number* of comparisons, then

$$c_n = (n-1) + \frac{1}{n} \sum_{k=1}^{n} (c_{k-1} + c_{n-k})$$

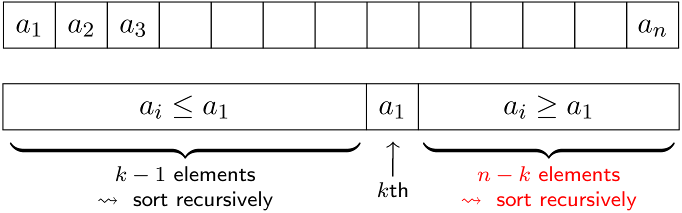
Recall: Quicksort



If c_n is the *average number* of comparisons, then

$$c_n = (n-1) + \frac{1}{n} \sum_{k=1}^{n} \left(c_{k-1} + c_{n-k} \right)$$

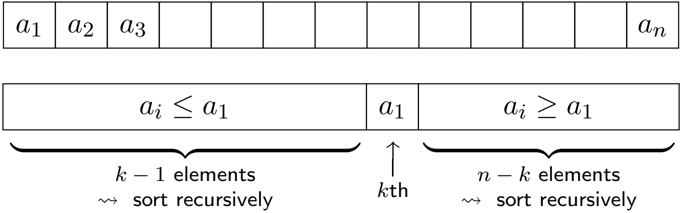
Recall: Quicksort



If c_n is the *average number* of comparisons, then

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}(c_{k-1} + c_{n-k})$$

Recall: Quicksort



If c_n is the *average number* of comparisons, then

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right) \qquad c_0 = 0$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}(c_{k-1} + c_{n-k})$$

$$0,\ 0,\ 1,\ \tfrac{8}{3},\ \tfrac{29}{6},\ \tfrac{37}{5},\ \tfrac{103}{10},\ \tfrac{472}{35},\ \tfrac{2369}{140},\ \tfrac{2593}{126},\ \tfrac{30791}{1260},\ \tfrac{32891}{1155},\ \tfrac{452993}{13860},\ \tfrac{476753}{12870},\ \tfrac{499061}{12012},$$

$$\frac{2080328}{45045},\ \frac{18358463}{360360},\ \frac{18999103}{340340},\ \frac{124184839}{2042040},\ \frac{127860511}{1939938},\ \frac{26274175}{369512},\ \frac{8982005}{117572},\ \frac{211524139}{2586584},$$

$$\frac{648798629}{7436429},\ \frac{16562041459}{178474296},\ \frac{16891532467}{171609900},\ \frac{154883957203}{1487285800},\ \frac{157646059403}{1434168450},\ \frac{4649180818987}{40156716600},$$

$$\frac{4724140023307}{38818159380},\ \frac{148699793966557}{1164544781400},\ \frac{603533261726728}{4512611027925},\ \frac{306005750313839}{2187932619600},\ \frac{28193110155949}{193052878200},$$

$$\frac{28557152726269}{187537081680},\ \frac{28911389436109}{182327718300},\ \frac{1082484349417033}{6563797858800},\ \frac{1094921019044233}{6391066336200},$$

$$\frac{1107047657733433}{6227192840400},\ \frac{1118879324130193}{6071513019390},\ \frac{46347630304850333}{242860520775600},\ \frac{46810221772994333}{237078127423800},$$

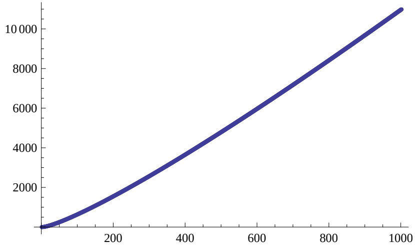
$$\frac{290325706098215417}{1422468764542800},\ \frac{3223454611135768387}{15291539218835100},\ \frac{3252678441642875467}{14951727236194320},\ \frac{3281281745920812427}{14626689687581400},$$

$$\frac{155536644130160510069}{672827725628744400},$$

$$\frac{155536644130160510069}{672827725628744400},\ \frac{156826230604282270169}{658810481344812225},\ \frac{7746413484856243587431}{31622903104550986800},$$

$$\frac{7807129458816981482087}{30990445042459967064},\ \frac{7866679725761316320759}{30382789257313693200},$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}(c_{k-1} + c_{n-k})$$



$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}(c_{k-1} + c_{n-k})$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n$$

c_n

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n \qquad\qquad c_n x^n$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n \qquad\qquad \sum_{n=0}^{\infty} c_n x^n$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n$$

$$\sum_{n=0}^{\infty} c_n x^n = -\frac{2x+2\log(1-x)}{(1-x)^2}$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n$$

$$\sum_{n=0}^{\infty} c_n x^n = -\frac{2x + 2\log(1-x)}{(1-x)^2}$$

$$c_n = 2(n+1)\sum_{k=0}^{n-1}\frac{k}{(k+1)(k+2)}$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n$$

$$\sum_{n=0}^{\infty} c_n x^n = -\frac{2x+2\log(1-x)}{(1-x)^2}$$

$$c_n = 2(n+1)\sum_{k=0}^{n-1}\frac{k}{(k+1)(k+2)}$$

$$c_n = -2n + 2\sum_{k=0}^{n} H_k$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n$$

$$\sum_{n=0}^{\infty} c_n x^n = -\frac{2x + 2\log(1-x)}{(1-x)^2}$$

$$c_n = 2(n+1)\sum_{k=0}^{n-1}\frac{k}{(k+1)(k+2)}$$

$$c_n = -2n + 2\sum_{k=0}^{n} H_k$$

$$c_n = 2(n+1)H_n - 4n$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(c_{k-1} + c_{n-k}\right)$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n$$

$$\sum_{n=0}^{\infty} c_n x^n = -\frac{2x+2\log(1-x)}{(1-x)^2}$$

$$c_n = 2(n+1)\sum_{k=0}^{n-1}\frac{k}{(k+1)(k+2)}$$

$$c_n = -2n + 2\sum_{k=0}^{n} H_k$$

$$c_n \sim 2n\log(n) \quad (n \to \infty)$$

$$c_n = 2(n+1)H_n - 4n$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}(c_{k-1} + c_{n-k})$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n$$

$$\sum_{n=0}^{\infty} c_n x^n = -\frac{2x+2\log(1-x)}{(1-x)^2}$$

$$c_n = 2(n+1)\sum_{k=0}^{n-1}\frac{k}{(k+1)(k+2)}$$

$$c_n = -2n + 2\sum_{k=0}^{n} H_k$$

$$c_n \sim 2n\log(n) \quad (n \to \infty)$$

$$c_n = 2(n+1)H_n - 4n$$

$$c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\bigl(c_{k-1} + c_{n-k}\bigr)$$

$$(n+1)c_{n+1} - (n+2)c_n = 2n$$

$$\sum_{n=0}^{\infty} c_n x^n = -\frac{2x+2\log(1-x)}{(1-x)^2}$$

$$c_n = 2(n+1)\sum_{k=0}^{n-1}\frac{k}{(k+1)(k+2)}$$

$$c_n = -2n + 2\sum_{k=0}^{n} H_k$$

$$c_n \sim 2n\log(n) \quad (n\to\infty)$$

$$c_n = 2(n+1)H_n - 4n$$

How to do such conversions using computer algebra.

How to do such conversions using computer algebra.

More precisely: We want algorithms for working with

How to do such conversions using computer algebra.

More precisely: We want algorithms for working with

- Symbolic sums

•

How to do such conversions using computer algebra.

More precisely: We want algorithms for working with

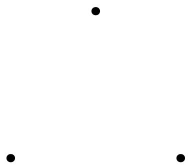
- Symbolic sums
- Recurrence equations

• •

How to do such conversions using computer algebra.

More precisely: We want algorithms for working with

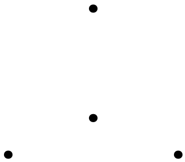
- Symbolic sums
- Recurrence equations
- Generating functions



How to do such conversions using computer algebra.

More precisely: We want algorithms for working with

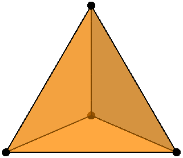
- Symbolic sums
- Recurrence equations
- Generating functions
- Asymptotic estimates



How to do such conversions using computer algebra.

More precisely: We want algorithms for working with

- Symbolic sums
- Recurrence equations
- Generating functions
- Asymptotic estimates

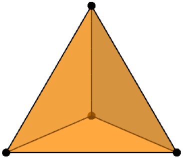


The interrelations between these four concepts form what we call *the concrete tetrahedron.*

How to do such conversions using computer algebra.

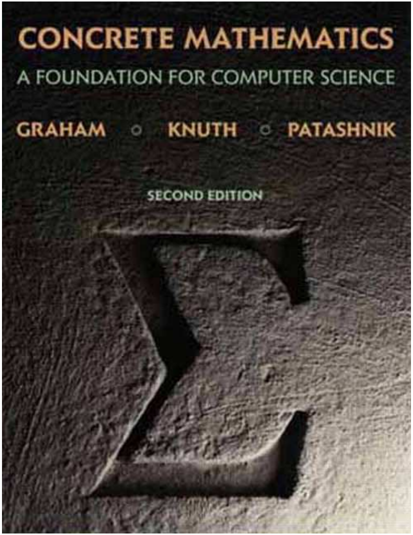
More precisely: We want algorithms for working with

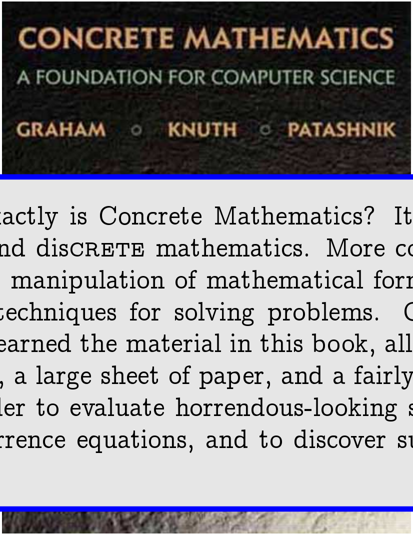
- Symbolic sums
- Recurrence equations
- Generating functions
- Asymptotic estimates



The interrelations between these four concepts form what we call *the concrete tetrahedron.*

Why *"concrete"*?


CONCRETE MATHEMATICS
A FOUNDATION FOR COMPUTER SCIENCE
GRAHAM ◦ KNUTH ◦ PATASHNIK
SECOND EDITION



"But what exactly is Concrete Mathematics? It is a blend of CONtinuous and disCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using a collection of techniques for solving problems. Once you, the reader, have learned the material in this book, all you will need is a cool head, a large sheet of paper, and a fairly decent handwriting in order to evaluate horrendous-looking sums, to solve complex recurrence equations, and to discover subtle patterns in data."

Let's agree for now on a slightly modified version:

> "But what exactly is Concrete Mathematics? It is a blend of CONtinuous and disCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using a collection of techniques for solving problems. Once you, the reader, have learned the material in this book, all you will need is a cool head, a large sheet of paper, and a fairly decent handwriting in order to evaluate horrendous-looking sums, to solve complex recurrence equations, and to discover subtle patterns in data."

Let's agree for now on a slightly modified version:

> "But what exactly is Concrete Mathematics? It is a blend of CONtinuous and disCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using ~~a collection of techniques~~ *symbolic computation* for solving problems. Once you, the reader, have learned the material in this book, all you will need is a cool head, a large sheet of paper, and a fairly decent handwriting in order to evaluate horrendous-looking sums, to solve complex recurrence equations, and to discover subtle patterns in data."

Let's agree for now on a slightly modified version:

> "But what exactly is Concrete Mathematics? It is a blend of CONtinuous and disCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using ~~a collection of techniques~~ *symbolic computation* for solving problems. Once you, the ~~reader~~ *student*, have learned the material in this book, all you will need is a cool head, a large sheet of paper, and a fairly decent handwriting in order to evaluate horrendous-looking sums, to solve complex recurrence equations, and to discover subtle patterns in data."

Let's agree for now on a slightly modified version:

> "But what exactly is Concrete Mathematics? It is a blend of CONtinuous and disCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using ~~a collection of techniques~~ *symbolic computation* for solving problems. Once you, the ~~reader~~ *student*, have learned the material in this ~~book~~ *lecture*, all you will need is a cool head, a large sheet of paper, and a fairly decent handwriting in order to evaluate horrendous-looking sums, to solve complex recurrence equations, and to discover subtle patterns in data."

Let's agree for now on a slightly modified version:

> "But what exactly is Concrete Mathematics? It is a blend of CONtinuous and disCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using ~~a collection of techniques~~ *symbolic computation* for solving problems. Once you, the ~~reader~~ *student*, have learned the material in this ~~book~~ *lecture*, all you will need is a ~~cool head, a large sheet of paper, and a fairly decent handwriting~~ *decent computer algebra system* in order to evaluate horrendous-looking sums, to solve complex recurrence equations, and to discover subtle patterns in data."

Naive wish: We want to have algorithms, which, given some infinite sequence $(a_n)_{n=0}^{\infty}$ as input,

Naive wish: We want to have algorithms, which, given some infinite sequence $(a_n)_{n=0}^{\infty}$ as input,

- compute the sequence of its partial sums: $b_n = \sum_{k=0}^{n} a_k$

Naive wish: We want to have algorithms, which, given some infinite sequence $(a_n)_{n=0}^{\infty}$ as input,

- compute the sequence of its partial sums: $b_n = \sum_{k=0}^{n} a_k$
- find its recurrence equations: $a_n = F(a_{n-1}, a_{n-2})$

Naive wish: We want to have algorithms, which, given some infinite sequence $(a_n)_{n=0}^{\infty}$ as input,

- compute the sequence of its partial sums: $b_n = \sum_{k=0}^{n} a_k$
- find its recurrence equations: $a_n = F(a_{n-1}, a_{n-2})$
- determine its generating function: $a(x) = \sum_{n=0}^{\infty} a_n x^n$

Naive wish: We want to have algorithms, which, given some infinite sequence $(a_n)_{n=0}^{\infty}$ as input,

- compute the sequence of its partial sums: $b_n = \sum_{k=0}^{n} a_k$
- find its recurrence equations: $a_n = F(a_{n-1}, a_{n-2})$
- determine its generating function: $a(x) = \sum_{n=0}^{\infty} a_n x^n$
- estimate its asymptotic growth: $a_n = \mathrm{O}(b_n)$ $(n \to \infty)$

Naive wish: We want to have algorithms, which, given some infinite sequence $(a_n)_{n=0}^{\infty}$ as input,

- compute the sequence of its partial sums: $b_n = \sum_{k=0}^{n} a_k$
- find its recurrence equations: $a_n = F(a_{n-1}, a_{n-2})$
- determine its generating function: $a(x) = \sum_{n=0}^{\infty} a_n x^n$
- estimate its asymptotic growth: $a_n = \mathrm{O}(b_n)$ $(n \to \infty)$

Problem: Such algorithms cannot exist.

Problem: Such algorithms cannot exist.

Problem: Such algorithms cannot exist.

Reason:

Problem: Such algorithms cannot exist.

Reason:

- Algorithms can only operate with *finite data.*

Problem: Such algorithms cannot exist.

Reason:

- Algorithms can only operate with *finite data.*
- There are only *countably many* different pieces of finite data.

Problem: Such algorithms cannot exist.

Reason:

- Algorithms can only operate with *finite data.*
- There are only *countably many* different pieces of finite data.
- But there are *uncountably many* infinite sequences.

Problem: Such algorithms cannot exist.

Reason:

- Algorithms can only operate with *finite data.*
- There are only *countably many* different pieces of finite data.
- But there are *uncountably many* infinite sequences.
- Hence there is *no data structure* for storing infinite sequences.

Problem: Such algorithms cannot exist.

Reason:

- Algorithms can only operate with *finite data.*
- There are only *countably many* different pieces of finite data.
- But there are *uncountably many* infinite sequences.
- Hence there is *no data structure* for storing infinite sequences.
- Hence there is *no algorithm* operating on infinite sequences.

Problem: Such algorithms cannot exist.

Reason:

- Algorithms can only operate with *finite data.*
- There are only *countably many* different pieces of finite data.
- But there are *uncountably many* infinite sequences.
- Hence there is *no data structure* for storing infinite sequences.
- Hence there is *no algorithm* operating on infinite sequences.

Workaround: Be more modest!

Problem: Such algorithms cannot exist.

Reason:

- Algorithms can only operate with *finite data.*
- There are only *countably many* different pieces of finite data.
- But there are *uncountably many* infinite sequences.
- Hence there is *no data structure* for storing infinite sequences.
- Hence there is *no algorithm* operating on infinite sequences.

Workaround: Be more modest!

Consider algorithms applicable to *certain* infinite sequences only.

Problem: Such algorithms cannot exist.

Reason:

- Algorithms can only operate with *finite data.*
- There are only *countably many* different pieces of finite data.
- But there are *uncountably many* infinite sequences.
- Hence there is *no data structure* for storing infinite sequences.
- Hence there is *no algorithm* operating on infinite sequences.

Workaround: Be more modest!

Consider algorithms applicable to *certain* infinite sequences only.
(For suitably chosen meanings of "certain".)

In other words:

In other words: We must limit ourselves to *classes of sequences,* which are such that their elements can be represented (distinguished) *in finite terms.*

In other words: We must limit ourselves to *classes of sequences,* which are such that their elements can be represented (distinguished) *in finite terms.*

This still leaves a lot freedom.

In other words: We must limit ourselves to *classes of sequences,* which are such that their elements can be represented (distinguished) *in finite terms.*

This still leaves a lot freedom.

What do we want from a such a class?

In other words: We must limit ourselves to *classes of sequences,* which are such that their elements can be represented (distinguished) *in finite terms.*

This still leaves a lot freedom.

What do we want from a such a class?

- *It should not be too big,* because the more special the elements in the class, the better we can compute with them.

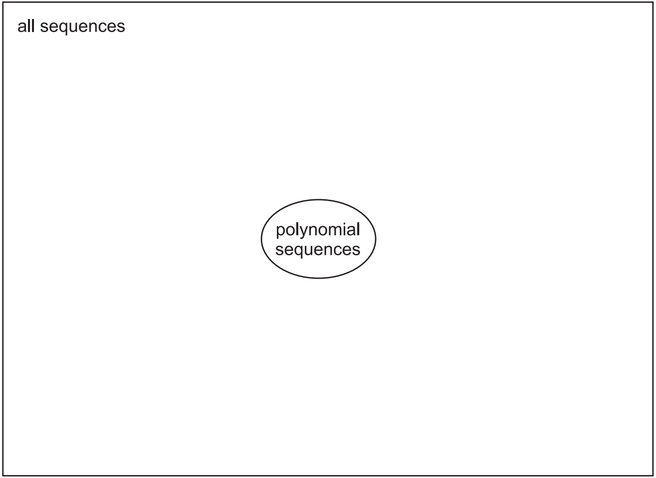
In other words: We must limit ourselves to *classes of sequences,* which are such that their elements can be represented (distinguished) *in finite terms.*

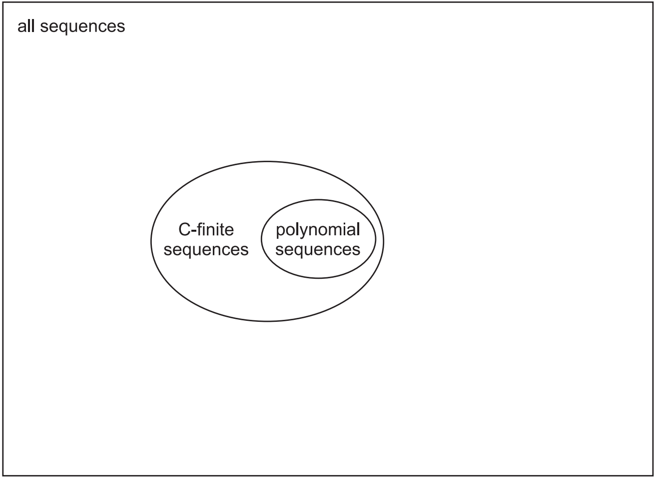
This still leaves a lot freedom.

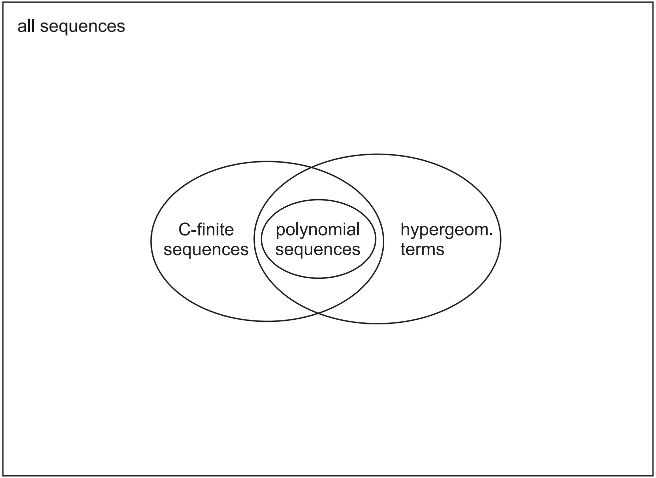
What do we want from a such a class?

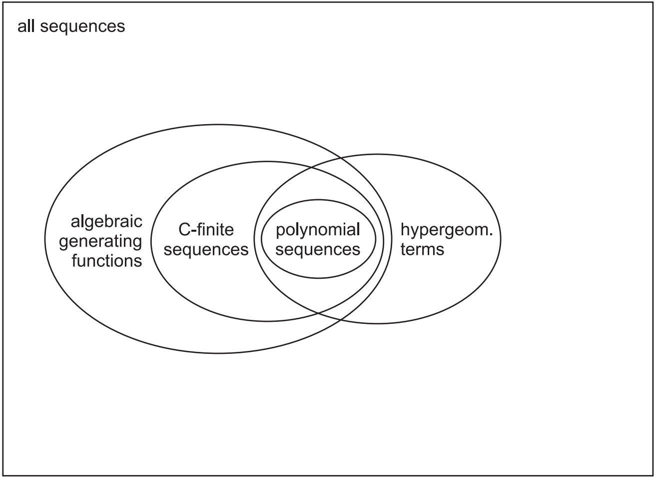
- *It should not be too big,* because the more special the elements in the class, the better we can compute with them.
- *It should not be too small,* because it should contain many sequences which arise in applications.

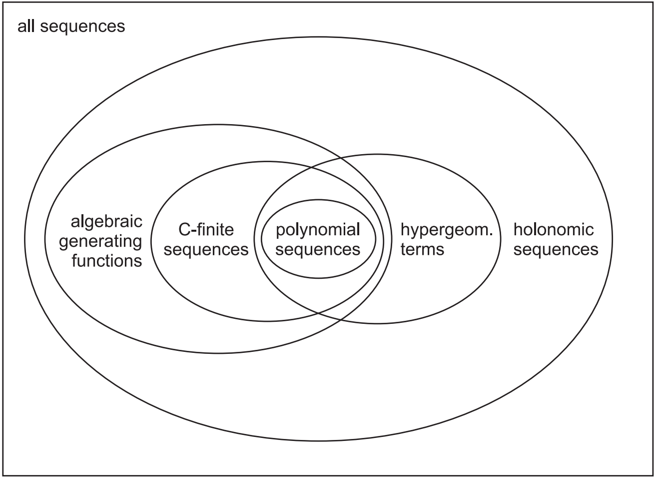

all sequences

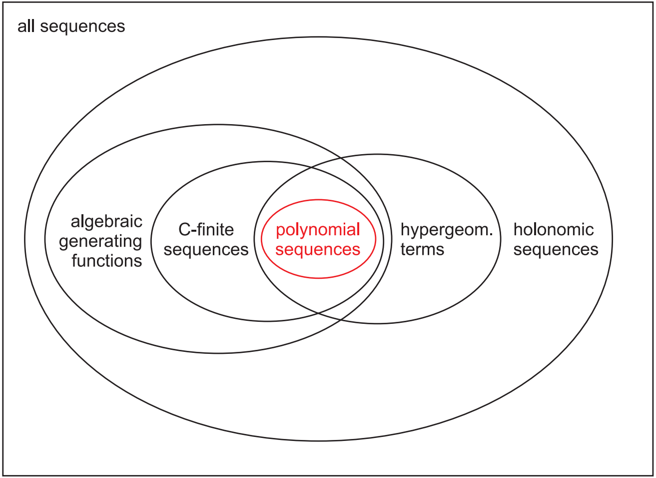

all sequences
polynomial sequences

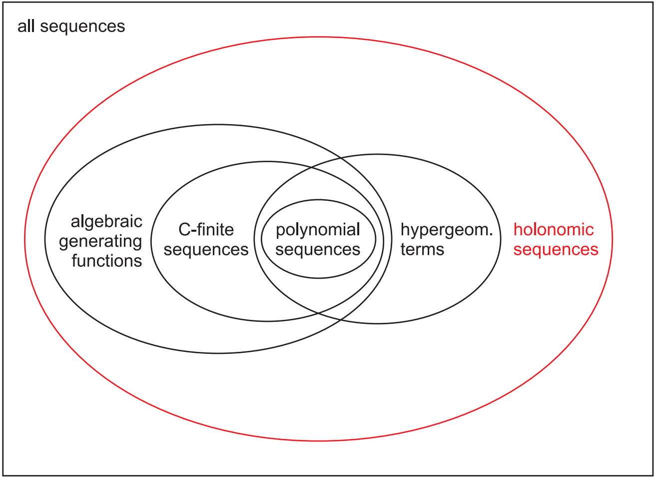

all sequences
C-finite sequences
polynomial sequences


all sequences
C-finite sequences
polynomial sequences
hypergeom. terms


all sequences
algebraic
generating
functions
C-finite
sequences
polynomial
sequences
hypergeom.
terms


all sequences
algebraic generating functions
C-finite sequences
polynomial sequences
hypergeom. terms
holonomic sequences


all sequences
algebraic generating functions
C-finite sequences
polynomial sequences
hypergeom. terms
holonomic sequences


all sequences
algebraic
generating
functions
C-finite
sequences
polynomial
sequences
hypergeom.
terms
holonomic
sequences

Summary:

Summary:

- We want to solve problems in discrete mathematics using computer algebra.

Summary:

- We want to solve problems in discrete mathematics using computer algebra.
- More precisely: We want to prove, discover, or simplify statements about infinite sequences.

Summary:

- We want to solve problems in discrete mathematics using computer algebra.
- More precisely: We want to prove, discover, or simplify statements about infinite sequences.
- The concrete tetrahedron:
    - Symbolic sums
    - Recurrence equations
    - Generating functions
    - Asymptotic estimates

Summary:

- We want to solve problems in discrete mathematics using computer algebra.
- More precisely: We want to prove, discover, or simplify statements about infinite sequences.
- The concrete tetrahedron:
  - Symbolic sums
  - Recurrence equations
  - Generating functions
  - Asymptotic estimates
- Classes of infinite sequences:
  - Polynomial sequences
  - C-finite sequences
  - Hypergeometric terms
  - Algebraic generating functions
  - Holonomic sequences

Polynomial Sequences

Defining property: A sequence $(a_n)_{n=0}^{\infty}$ is a *polynomial sequence* if there exists a polynomial p such that $a_n = p(n)$ for all $n \geq 0$.

Defining property: A sequence $(a_n)_{n=0}^{\infty}$ is a *polynomial sequence* if there exists a polynomial p such that $a_n = p(n)$ for all $n \geq 0$.

(Don't confuse with sequences of polynomials!)

Defining property: A sequence $(a_n)_{n=0}^{\infty}$ is a *polynomial sequence* if there exists a polynomial p such that $a_n = p(n)$ for all $n \geq 0$.

(Don't confuse with sequences of polynomials!)

Examples:

- $a_n = n^6 - 7n^5 + 108n^4 - 23n^3 + \frac{432}{309}n^2 + 349n - 1923478$
- $a_n = (n-1)^{30}$
- $a_n =$ number of 3×3 magic squares with magic constant n

Defining property: A sequence $(a_n)_{n=0}^{\infty}$ is a *polynomial sequence* if there exists a polynomial p such that $a_n = p(n)$ for all $n \geq 0$.

(Don't confuse with sequences of polynomials!)

Examples:

- $a_n = n^6 - 7n^5 + 108n^4 - 23n^3 + \frac{432}{309}n^2 + 349n - 1923478$
- $a_n = (n-1)^{30}$
- $a_n =$ number of 3×3 magic squares with magic constant n
  $= \frac{1}{8}(n+1)(n+2)(n^2+3n+4)$

Some ways of representing polynomial sequences “in finite terms”:

Some ways of representing polynomial sequences “in finite terms”:

- By the coefficient list (“in closed form”)

  Example: $a_n = 3n^2 - 4n + 2$

Some ways of representing polynomial sequences "in finite terms":

- By the coefficient list ("in closed form")

  Example: $a_n = 3n^2 - 4n + 2$

- By the coefficient list with respect to some special basis

  Example: $a_n = 6\binom{n}{2} - \binom{n}{1} + 2\binom{n}{0}$

Some ways of representing polynomial sequences "in finite terms":

- By the coefficient list ("in closed form")

  Example: $a_n = 3n^2 - 4n + 2$

- By the coefficient list with respect to some special basis

  Example: $a_n = 6\binom{n}{2} - \binom{n}{1} + 2\binom{n}{0}$

- By recurrence and initial values

  Example: $a_{n+3} = a_n - 3a_{n+1} + 3a_{n+2},\ \ a_0{=}2,\ \ a_1{=}1,\ \ a_2{=}6.$

Some ways of representing polynomial sequences "in finite terms":

- By the coefficient list ("in closed form")

  Example: $a_n = 3n^2 - 4n + 2$

- By the coefficient list with respect to some special basis

  Example: $a_n = 6\binom{n}{2} - \binom{n}{1} + 2\binom{n}{0}$

- By recurrence and initial values

  Example: $a_{n+3} = a_n - 3a_{n+1} + 3a_{n+2}, \;\; a_0{=}2, \;\; a_1{=}1, \;\; a_2{=}6.$

- By its generating function ("in closed form")

  Example: $\sum_{n=0}^{\infty} a_n x^n = \frac{9x^2-5x+2}{(1-x)^3}$

A Conversion

A Conversion

- *closed form → recurrence and initial values:*

A Conversion

- *closed form → recurrence and initial values:*
  Easy: initial values by evaluation, and the recurrence for a polynomial sequence of degree d is always

$$a_n - (d+1)a_{n+1} + \binom{d+1}{2}a_{n+2} - \binom{d+1}{3}a_{n+3} \pm \cdots + (-1)^i\binom{d+1}{i}a_{n+i} \pm \cdots + (-1)^{d+1}a_{n+d+1} = 0.$$

A Conversion

- *closed form → recurrence and initial values:*
  Easy: initial values by evaluation, and the recurrence for a polynomial sequence of degree d is always

$$a_n - (d+1)a_{n+1} + \binom{d+1}{2}a_{n+2} - \binom{d+1}{3}a_{n+3} \pm \cdots + (-1)^i\binom{d+1}{i}a_{n+i} \pm \cdots + (-1)^{d+1}a_{n+d+1} = 0.$$

- *recurrence and initial values → closed form:*

A Conversion

- *closed form $\to$ recurrence and initial values:*
  Easy: initial values by evaluation, and the recurrence for a polynomial sequence of degree d is always

$$a_n - (d+1)a_{n+1} + \binom{d+1}{2}a_{n+2} - \binom{d+1}{3}a_{n+3} \pm \cdots + (-1)^i\binom{d+1}{i}a_{n+i} \pm \cdots + (-1)^{d+1}a_{n+d+1} = 0.$$

- *recurrence and initial values $\to$ closed form:*
  Also easy: interpolation of initial values.

A Conversion

- *closed form → generating function:*

A Conversion

- *closed form $\rightarrow$ generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

A Conversion

- *closed form $\rightarrow$ generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \Big| \ \frac{d}{dx}$$

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} x^n = \tfrac{1}{1-x} \qquad \Big| \; \tfrac{d}{dx} \quad \Big| \; \cdot x$$

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \Big| \; \frac{d}{dx} \; \Big| \cdot x$$

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \Big| \ \frac{d}{dx} \ \Big| \cdot x$$

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad \Big| \ \frac{d}{dx} \ \Big| \cdot x$$

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \Big| \frac{d}{dx} \quad \Big| \cdot x$$

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad \Big| \frac{d}{dx} \quad \Big| \cdot x$$

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$$

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \Big| \ \frac{d}{dx} \ \Big| \ \cdot x$$

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad \Big| \ \frac{d}{dx} \ \Big| \ \cdot x$$

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3} \qquad \Big| \ \frac{d}{dx} \ \Big| \ \cdot x$$

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n &= \tfrac{1}{1-x} && \Big| \tfrac{d}{dx} \quad \Big| \cdot x \\
\sum_{n=0}^{\infty} n x^n &= \tfrac{x}{(1-x)^2} && \Big| \tfrac{d}{dx} \quad \Big| \cdot x \\
\sum_{n=0}^{\infty} n^2 x^n &= \tfrac{x(x+1)}{(1-x)^3} && \Big| \tfrac{d}{dx} \quad \Big| \cdot x \\
\sum_{n=0}^{\infty} n^3 x^n &= \tfrac{x(x^2+4x+1)}{(1-x)^4} && \dots
\end{aligned}$$

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$5 - 3n + n^2 + 2n^3$$

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} \left(5 - 3n + n^2 + 2n^3\right) x^n$$

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty}\left(5-3n+n^2+2n^3\right)x^n$$
$$=5\tfrac{1}{1-x}-3\tfrac{x}{(1-x)^2}+\tfrac{x(x+1)}{(1-x)^3}+2\tfrac{x(x^2+4x+1)}{(1-x)^4}$$

A Conversion

- *closed form $\rightarrow$ generating function:*
  Use the geometric series and its derivatives:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(5-3n+n^2+2n^3\right)x^n \\
&= 5\tfrac{1}{1-x} - 3\tfrac{x}{(1-x)^2} + \tfrac{x(x+1)}{(1-x)^3} + 2\tfrac{x(x^2+4x+1)}{(1-x)^4} \\
&= \tfrac{-7x^3+29x^2-15x+5}{(1-x)^4}.
\end{aligned}$$

A Conversion

- *closed form $\to$ generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} \left(5 - 3n + n^2 + 2n^3\right) x^n$$
$$= 5\tfrac{1}{1-x} - 3\tfrac{x}{(1-x)^2} + \tfrac{x(x+1)}{(1-x)^3} + 2\tfrac{x(x^2+4x+1)}{(1-x)^4}$$
$$= \tfrac{-7x^3+29x^2-15x+5}{(1-x)^4}.$$

- *generating function $\to$ closed form:*

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty}\left(5-3n+n^2+2n^3\right)x^n$$
$$= 5\tfrac{1}{1-x} - 3\tfrac{x}{(1-x)^2} + \tfrac{x(x+1)}{(1-x)^3} + 2\tfrac{x(x^2+4x+1)}{(1-x)^4}$$
$$= \tfrac{-7x^3+29x^2-15x+5}{(1-x)^4}.$$

- *generating function → closed form:*
  Easy: interpolate the first $d+1$ terms of the Taylor expansion.

A Conversion

- *closed form → generating function:*
  Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty}\left(5-3n+n^2+2n^3\right)x^n$$
$$=5\tfrac{1}{1-x}-3\tfrac{x}{(1-x)^2}+\tfrac{x(x+1)}{(1-x)^3}+2\tfrac{x(x^2+4x+1)}{(1-x)^4}$$
$$=\tfrac{-7x^3+29x^2-15x+5}{(1-x)^4}.$$

- *generating function → closed form:*
  Easy: interpolate the first $d+1$ terms of the Taylor expansion.
  Or: Ansatz and coefficient comparison.

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- It this a polynomial sequence?

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- It this a polynomial sequence?
- We can't tell for sure without knowing how it continues.

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- It this a polynomial sequence?
- We can't tell for sure without knowing how it continues.
- But it can be instructive to find plausible candidates.

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- It this a polynomial sequence?
- We can't tell for sure without knowing how it continues.
- But it can be instructive to find plausible candidates.
- Good candidates often give useful hints about the problem from which the sequence originates.

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- It this a polynomial sequence?
- We can't tell for sure without knowing how it continues.
- But it can be instructive to find plausible candidates.
- Good candidates often give useful hints about the problem from which the sequence originates.
- Once a conjecture is born, it may be possible to prove it by an independent argument.

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- It this a polynomial sequence?
- We can't tell for sure without knowing how it continues.
- But it can be instructive to find plausible candidates.
- Good candidates often give useful hints about the problem from which the sequence originates.
- Once a conjecture is born, it may be possible to prove it by an independent argument.
- How to find trustworthy candidates?

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- *Interpolation.*

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- *Interpolation.*

  If the interpolating polynomial of the first N terms has degree $d << N$, then this is a strong indication for a polynomial sequence.

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- *Interpolation.*
- *Pade Approximation.*

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- *Interpolation.*
- *Pade Approximation.*

  If the Pade approximant of the first N terms has the form $\frac{\mathrm{poly}(x)}{(1-x)^{d+1}}$, then this hints at a polynomial sequence of degree $\leq d$.

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- *Interpolation.*
- *Pade Approximation.*
- *Recurrence Matching.*

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- *Interpolation.*
- *Pade Approximation.*
- *Recurrence Matching.*
  If the given data matches the linear recurrence for polynomials of degree d, then this is perhaps not just a coincidence.

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- *Interpolation.*
- *Pade Approximation.*
- *Recurrence Matching.*
- *Asymptotics.*

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- *Interpolation.*
- *Pade Approximation.*
- *Recurrence Matching.*
- *Asymptotics.*

  If $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d, then $\lim_{n\to\infty} \frac{n(a_{n+1}-a_n)}{a_n} = d$.

B Guessing

$2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, \ldots$

- *Interpolation.*
- *Pade Approximation.*
- *Recurrence Matching.*
- *Asymptotics.*

  If $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d, then $\lim_{n\to\infty} \frac{n(a_{n+1}-a_n)}{a_n} = d$.

  Therefore, if $n(a_{n+1} - a_n)/a_n$ does not seem to converge to a nonnegative integer, our sequence is probably not polynomial.

C Asymptotics

C Asymptotics

- *From the closed form:* trivial.

C Asymptotics

- *From the closed form:* trivial.
- *From the generating function:*

C Asymptotics

- *From the closed form:* trivial.
- *From the generating function:* In general, the asymptotic behavior of any sequence $(a_n)_{n=0}^{\infty}$ is determined by the singularities of its generating function which are closest to 0

C Asymptotics

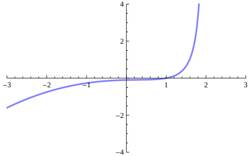
- *From the closed form:* trivial.
- *From the generating function:* In general, the asymptotic behavior of any sequence $(a_n)_{n=0}^{\infty}$ is determined by the singularities of its generating function which are closest to 0

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

C Asymptotics

- *From the closed form:* trivial.
- *From the generating function:* In general, the asymptotic behavior of any sequence $(a_n)_{n=0}^{\infty}$ is determined by the singularities of its generating function which are closest to 0

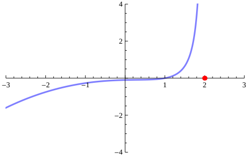
$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$



C *Asymptotics*

- *From the closed form:* trivial.
- *From the generating function:* In general, the asymptotic behavior of any sequence $(a_n)_{n=0}^{\infty}$ is determined by the singularities of its generating function which are closest to 0

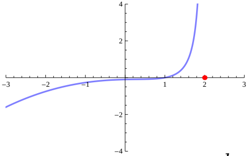
$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$



C Asymptotics

- *From the closed form:* trivial.
- *From the generating function:* In general, the asymptotic behavior of any sequence $(a_n)_{n=0}^{\infty}$ is determined by the singularities of its generating function which are closest to 0

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

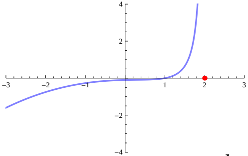


- A pole of multiplicity d at $x = \xi$ implies $a_n = \mathrm{O}(n^{d-1}\xi^{-n})$.

C *Asymptotics*

- *From the closed form:* trivial.
- *From the generating function:* In general, the asymptotic behavior of any sequence $(a_n)_{n=0}^{\infty}$ is determined by the singularities of its generating function which are closest to 0

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$



- A pole of multiplicity d at $x = \xi$ implies $a_n = \mathrm{O}(n^{d-1}\xi^{-n})$.
- For polynomial sequences of degree d, it follows $a_n = \mathrm{O}(n^d)$.

D Summation

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

- Define $n^{\underline{d}} := n(n-1)(n-2)\cdots(n-d+1)$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

- Define $n^{\underline{d}} := n(n-1)(n-2)\cdots(n-d+1)$
- Then $1, n, n^{\underline{2}}, n^{\underline{3}}, n^{\underline{3}}, \dots$ is a vector space basis of $K[n]$.

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

- Define $n^{\underline{d}} := n(n-1)(n-2)\cdots(n-d+1)$
- Then $1, n, n^{\underline{2}}, n^{\underline{3}}, n^{\underline{3}}, \ldots$ is a vector space basis of $K[n]$.
- Polynomials expressed in this basis can be summed termwise:

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

- Define $n^{\underline{d}} := n(n-1)(n-2)\cdots(n-d+1)$
- Then $1, n, n^{\underline{2}}, n^{\underline{3}}, n^{\underline{3}}, \dots$ is a vector space basis of $K[n]$.
- Polynomials expressed in this basis can be summed termwise:

$$k^3 + 4k - 7$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

- Define $n^{\underline{d}} := n(n-1)(n-2)\cdots(n-d+1)$
- Then $1, n, n^{\underline{2}}, n^{\underline{3}}, n^{\underline{3}}, \ldots$ is a vector space basis of $K[n]$.
- Polynomials expressed in this basis can be summed termwise:

$$k^3 + 4k - 7$$

$$\downarrow$$

$$k^{\underline{3}} + 3k^{\underline{2}} + 5k - 7$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

- Define $n^{\underline{d}} := n(n-1)(n-2)\cdots(n-d+1)$
- Then $1, n, n^{\underline{2}}, n^{\underline{3}}, n^{\underline{3}}, \ldots$ is a vector space basis of $K[n]$.
- Polynomials expressed in this basis can be summed termwise:

$$k^3 + 4k - 7$$

$$\downarrow$$

$$k^{\underline{3}} + 3k^{\underline{2}} + 5k - 7 \xrightarrow[\sum]{} \tfrac{1}{4}n^{\underline{4}} + \tfrac{3}{3}n^{\underline{3}} + \tfrac{5}{2}n^{\underline{2}} - 7n$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

- Define $n^{\underline{d}} := n(n-1)(n-2)\cdots(n-d+1)$
- Then $1, n, n^{\underline{2}}, n^{\underline{3}}, n^{\underline{3}}, \ldots$ is a vector space basis of $K[n]$.
- Polynomials expressed in this basis can be summed termwise:

$$\begin{array}{ccc} k^3 + 4k - 7 & & \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{9}{4}n^2 - 9n \\ \downarrow & & \uparrow \\ k^{\underline{3}} + 3k^{\underline{2}} + 5k - 7 & \xrightarrow[\sum]{} & \frac{1}{4}n^{\underline{4}} + \frac{3}{3}n^{\underline{3}} + \frac{5}{2}n^{\underline{2}} - 7n \end{array}$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

- Define $n^{\underline{d}} := n(n-1)(n-2)\cdots(n-d+1)$
- Then $1, n, n^{\underline{2}}, n^{\underline{3}}, n^{\underline{3}}, \dots$ is a vector space basis of $K[n]$.
- Polynomials expressed in this basis can be summed termwise:

$$\begin{array}{ccc}
k^3 + 4k - 7 & \xrightarrow{\ \sum\ } & \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{9}{4}n^2 - 9n \\
\downarrow & & \uparrow \\
k^{\underline{3}} + 3k^{\underline{2}} + 5k - 7 & \xrightarrow[\ \sum\ ]{} & \frac{1}{4}n^{\underline{4}} + \frac{3}{3}n^{\underline{3}} + \frac{5}{2}n^{\underline{2}} - 7n
\end{array}$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

1. via basis conversion

- Define $n^{\underline{d}} := n(n-1)(n-2)\cdots(n-d+1)$
- Then $1, n, n^{\underline{2}}, n^{\underline{3}}, n^{\underline{3}}, \dots$ is a vector space basis of $K[n]$.
- Polynomials expressed in this basis can be summed termwise
- Mnemonic:

$$\sum_{k=0}^{n-1} k^{\underline{d}} = \frac{1}{d+1} n^{\underline{d+1}} \quad \longleftrightarrow \quad \int_0^x t^d dt = \frac{1}{d+1} x^{d+1}$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

2. via the generating function

- Use the multiplication law for power series:

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

2. via the generating function

- Use the multiplication law for power series:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

2. via the generating function

- Use the multiplication law for power series:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n$$

- For $b_n = 1$ this turns into

$$\frac{1}{1-x}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k\right)x^n$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

2. via the generating function

- We can sum sequences by multiplying their generating function with $\frac{1}{1-x}$.

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

2. via the generating function

- We can sum sequences by multiplying their generating function with $\frac{1}{1-x}$.

$$k^3 + 4k - 7$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

2. via the generating function

- We can sum sequences by multiplying their generating function with $\frac{1}{1-x}$.

$$k^3 + 4k - 7$$

$$\Big\downarrow \text{gfun}$$

$$\frac{12x^3 - 25x^2 + 26x - 7}{(1-x)^4}$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

2. via the generating function

- We can sum sequences by multiplying their generating function with $\frac{1}{1-x}$.

$$k^3 + 4k - 7$$

$$\Big\downarrow \text{gfun}$$

$$\frac{12x^3 - 25x^2 + 26x - 7}{(1-x)^4} \xrightarrow[\cdot \frac{1}{1-x}]{} \frac{12x^3 - 25x^2 + 26x - 7}{(1-x)^{5}}$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

2. via the generating function

- We can sum sequences by multiplying their generating function with $\frac{1}{1-x}$.

$$
\begin{array}{ccc}
k^3 + 4k - 7 & & \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{9}{4}n^2 - 9n \\
\Big\downarrow \text{gfun} & & \Big\uparrow \text{un-gfun} \\
\dfrac{12x^3 - 25x^2 + 26x - 7}{(1-x)^4} & \xrightarrow[\cdot \frac{1}{1-x}]{} & \dfrac{12x^3 - 25x^2 + 26x - 7}{(1-x)^{5}}
\end{array}
$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

2. via the generating function

- We can sum sequences by multiplying their generating function with $\frac{1}{1-x}$.

$$\begin{array}{ccc} k^3+4k-7 & \xrightarrow{\sum} & \frac{1}{4}n^4-\frac{1}{2}n^3+\frac{9}{4}n^2-9n \\ \text{gfun}\downarrow & & \uparrow\text{un-gfun} \\ \dfrac{12x^3-25x^2+26x-7}{(1-x)^4} & \xrightarrow[\cdot\frac{1}{1-x}]{} & \dfrac{12x^3-25x^2+26x-7}{(1-x)^{5}} \end{array}$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

3. via the initial values

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

3. via the initial values

- Note: if $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d then $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is a polynomial sequence of degree $d+1$.

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

3. via the initial values

- Note: if $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d then $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is a polynomial sequence of degree $d+1$.
- As such, it is uniquely determined by its first $d+2$ values.

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

3. via the initial values

- Note: if $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d then $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is a polynomial sequence of degree $d+1$.
- As such, it is uniquely determined by its first $d+2$ values.
- We can find the sum via evaluation/interpolation:

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

3. via the initial values

- Note: if $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d then $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is a polynomial sequence of degree $d+1$.
- As such, it is uniquely determined by its first $d+2$ values.
- We can find the sum via evaluation/interpolation:

$$k^3 + 4k - 7$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

3. via the initial values

- Note: if $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d then $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is a polynomial sequence of degree $d+1$.
- As such, it is uniquely determined by its first $d+2$ values.
- We can find the sum via evaluation/interpolation:

$$k^3 + 4k - 7$$

evaluate ↓

$$-7, -2, 9, 32, 73$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

3. via the initial values

- Note: if $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d then $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is a polynomial sequence of degree $d+1$.
- As such, it is uniquely determined by its first $d+2$ values.
- We can find the sum via evaluation/interpolation:

$$k^3 + 4k - 7$$

evaluate ↓

$$-7, -2, 9, 32, 73 \xrightarrow{\sum} -7, -9, 0, 32, 105$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

3. via the initial values

- Note: if $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d then $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is a polynomial sequence of degree $d+1$.
- As such, it is uniquely determined by its first $d+2$ values.
- We can find the sum via evaluation/interpolation:

$$k^3 + 4k - 7 \qquad\qquad \tfrac{1}{4}n^4 - \tfrac{1}{2}n^3 + \tfrac{9}{4}n^2 - 9n$$

evaluate ↓ ↑ interpolate

$$-7, -2, 9, 32, 73 \xrightarrow{\;\sum\;} -7, -9, 0, 32, 105$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

3. via the initial values

- Note: if $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d then $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is a polynomial sequence of degree $d+1$.
- As such, it is uniquely determined by its first $d+2$ values.
- We can find the sum via evaluation/interpolation:

$$\begin{array}{ccc} k^3+4k-7 & \xrightarrow{\ \sum\ } & \frac{1}{4}n^4-\frac{1}{2}n^3+\frac{9}{4}n^2-9n \\ \text{evaluate}\ \big\downarrow & & \big\uparrow\ \text{interpolate} \\ -7,-2,9,32,73 & \xrightarrow{\ \sum\ } & -7,-9,0,32,105 \end{array}$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

4. via Faulhaber's formula

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

4. via Faulhaber's formula

- Let B_n denote the nth Bernoulli number.

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

4. via Faulhaber's formula

- Let B_n denote the nth Bernoulli number.
- Then

$$\sum_{k=0}^{n} k^d = \frac{1}{d+1} \sum_{k=0}^{d} B_k \binom{d+1}{k} (n+1)^{d-k+1}.$$

D Summation

Given a polynomial sequence $(a_n)_{n=0}^{\infty}$, find $\sum_{k=0}^{n} a_k$.

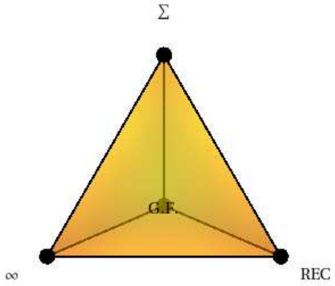
4. via Faulhaber's formula

- Let B_n denote the nth Bernoulli number.
- Then

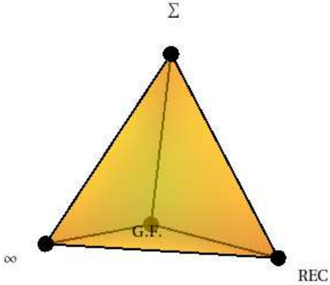
$$\sum_{k=0}^{n} k^d = \frac{1}{d+1} \sum_{k=0}^{d} B_k \binom{d+1}{k} (n+1)^{d-k+1}.$$

- This can be used to sum a polynomial termwise in the standard basis.

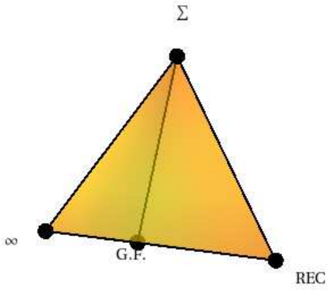
Summary.



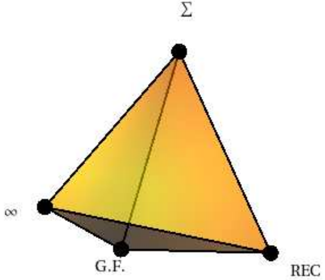
Summary.



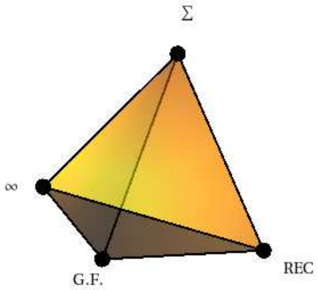
Summary.



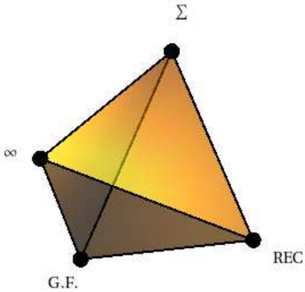
Summary.



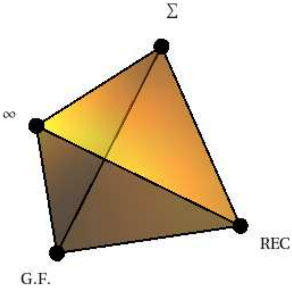
Summary.



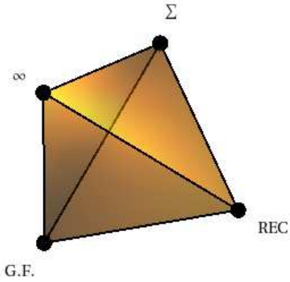
Summary.



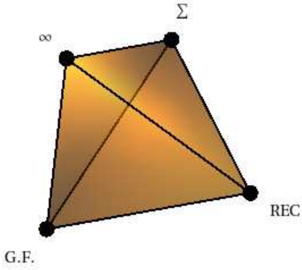
Summary.



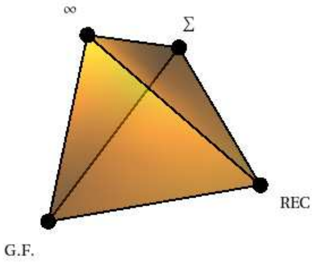
Summary.



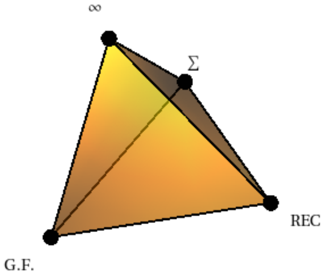
Summary.



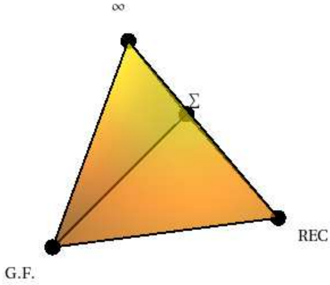
Summary.



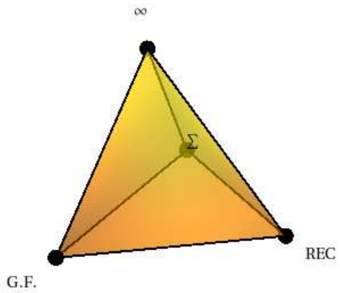
Summary.



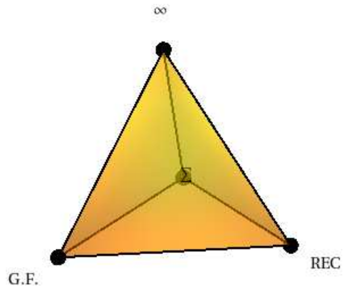
Summary.



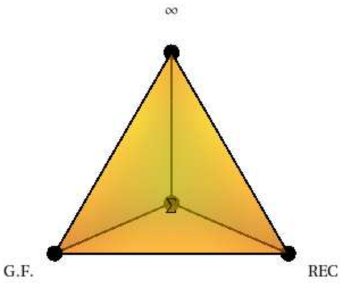
Summary.



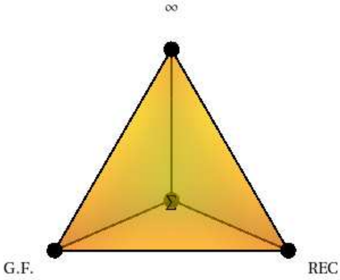
Summary.



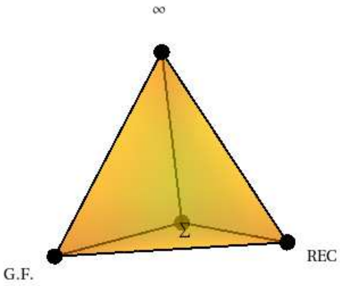
Summary.



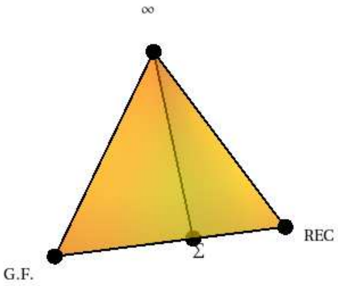
Summary.



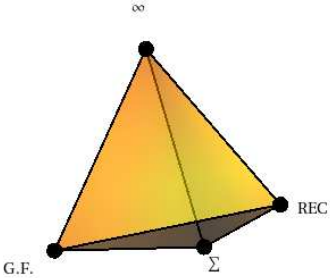
Summary.



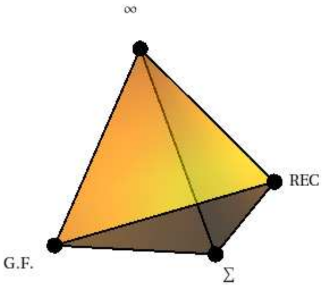
Summary.



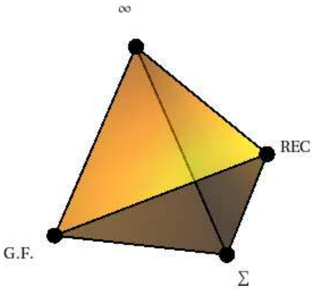
Summary.



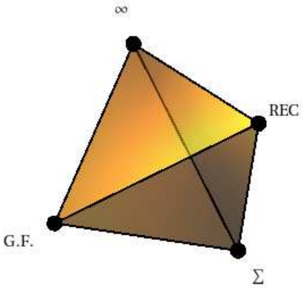
Summary.



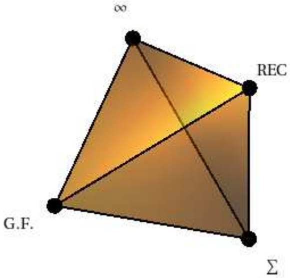
Summary.



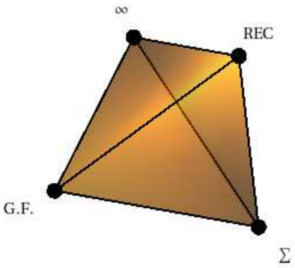
Summary.



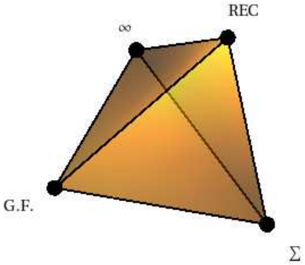
Summary.



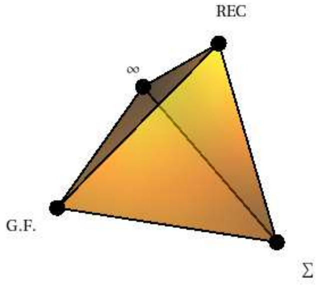
Summary.



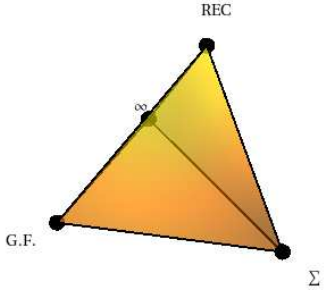
Summary.



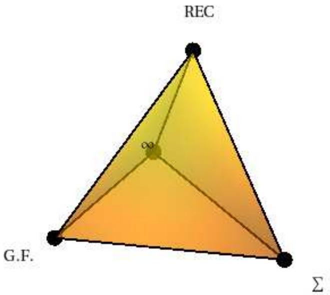
Summary.



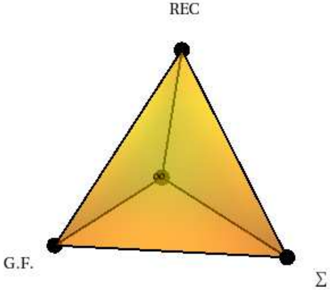
Summary.



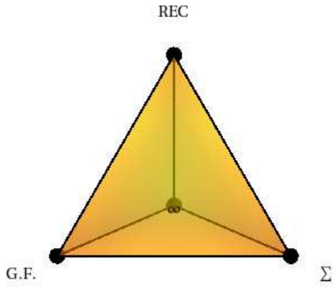
Summary.



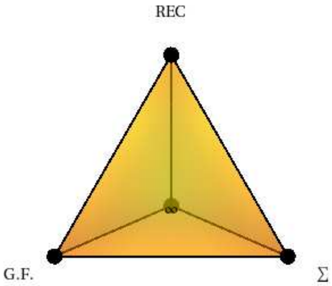
Summary.



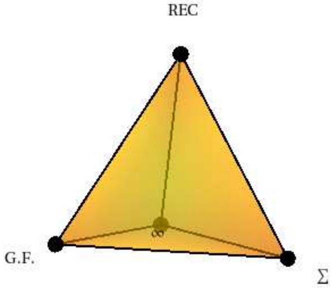
Summary.



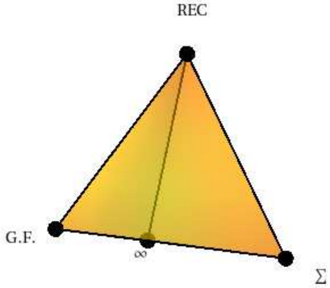
Summary.



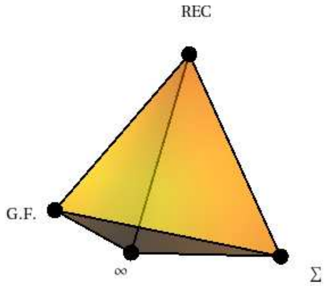
Summary.



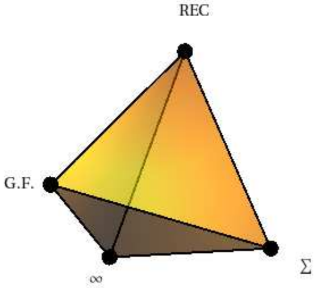
Summary.



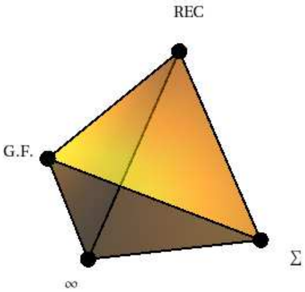
Summary.



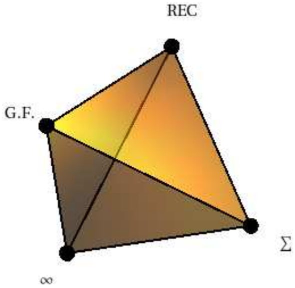
Summary.



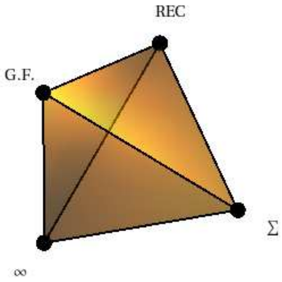
Summary.



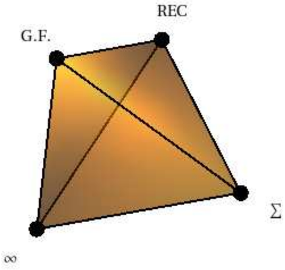
Summary.



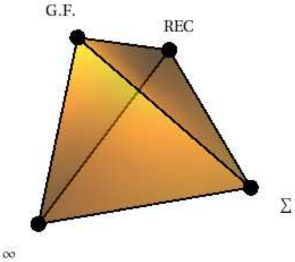
Summary.



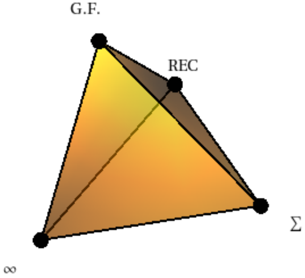
Summary.



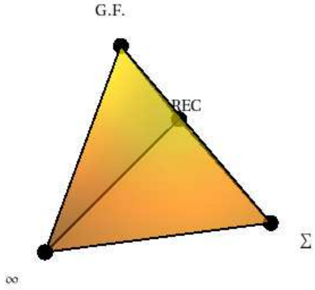
Summary.



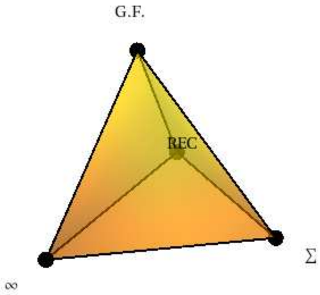
Summary.



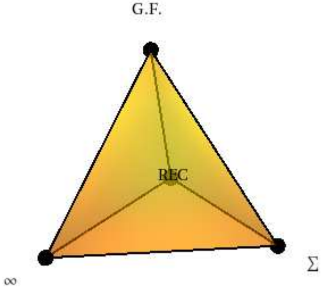
Summary.



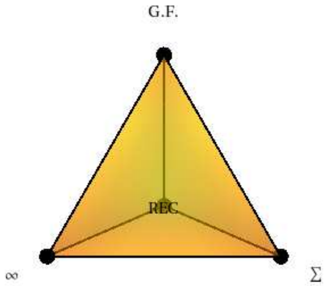
Summary.



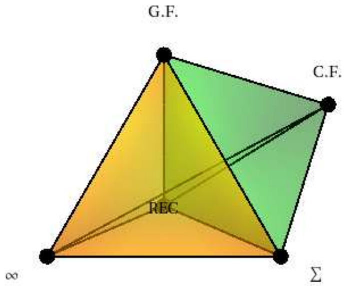
Summary.



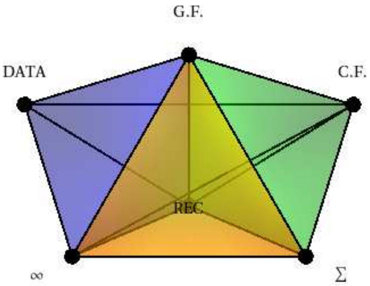
Summary.

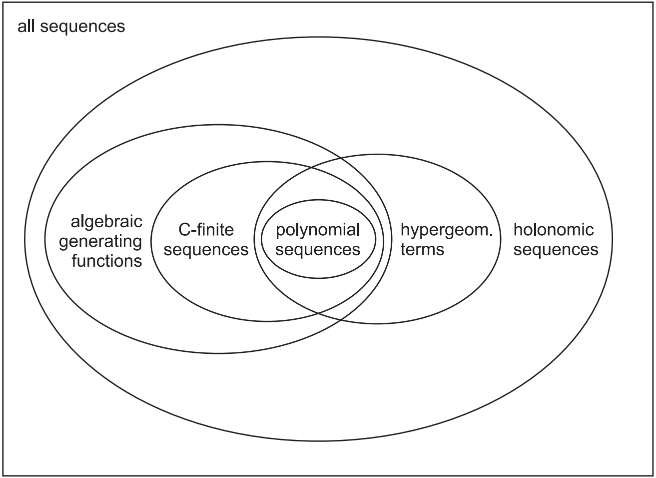


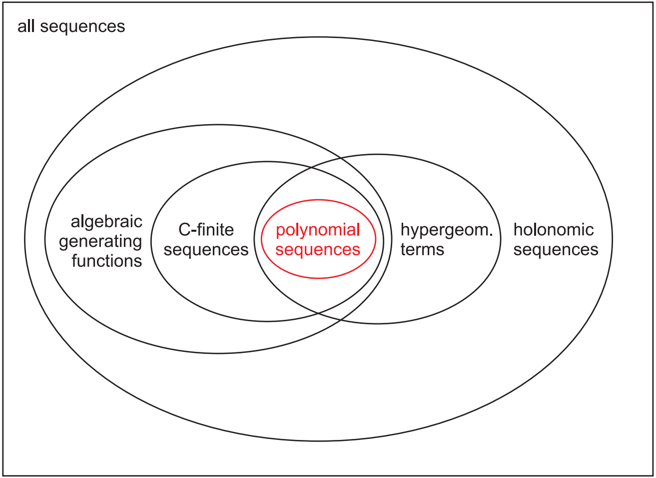
Summary.

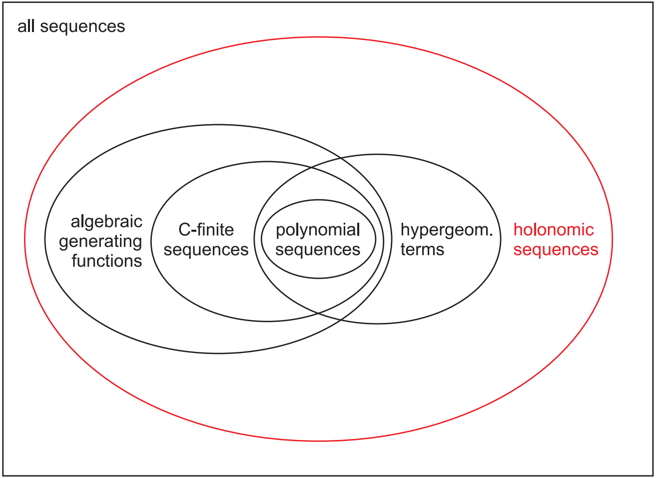


Summary.



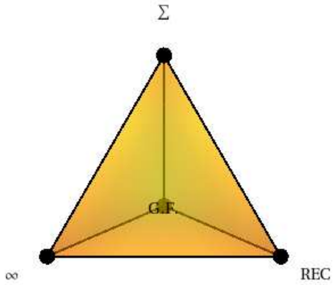

all sequences
algebraic
generating
functions
C-finite
sequences
polynomial
sequences
hypergeom.
terms
holonomic
sequences


all sequences
algebraic
generating
functions
C-finite
sequences
polynomial
sequences
hypergeom.
terms
holonomic
sequences

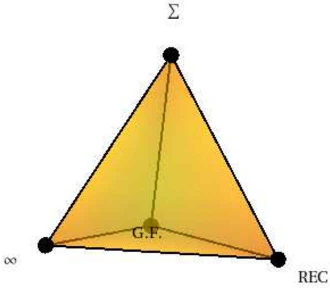

all sequences
algebraic generating functions
C-finite sequences
polynomial sequences
hypergeom. terms
holonomic sequences

Holonomic Sequences and Power Series

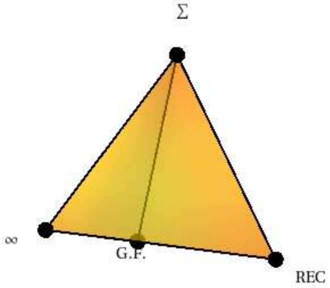
Recall:



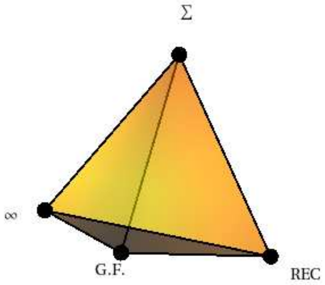
Recall:



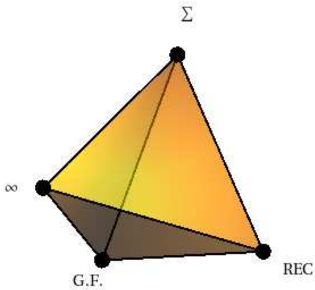
Recall:



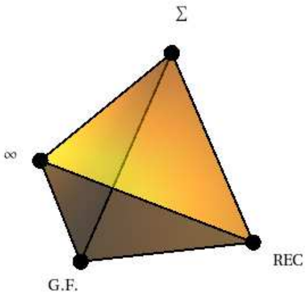
Recall:



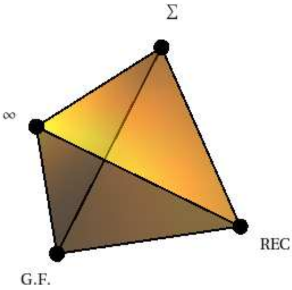
Recall:



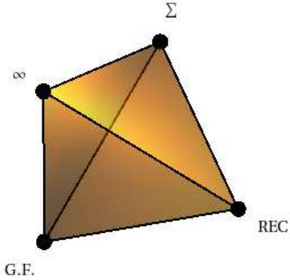
Recall:



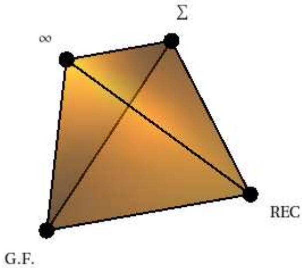
Recall:



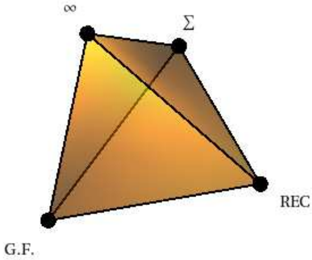
Recall:



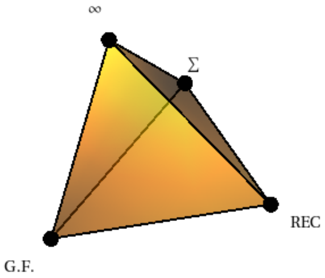
Recall:



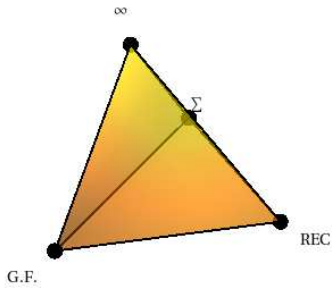
Recall:



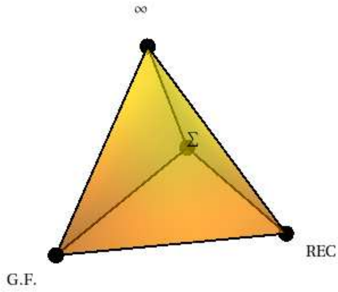
Recall:



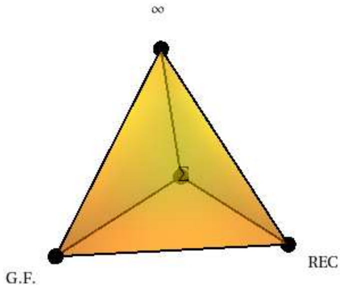
Recall:



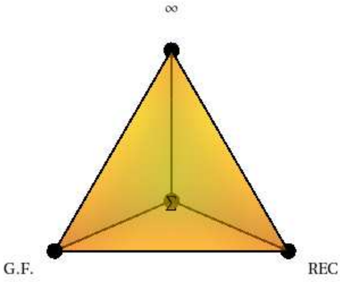
Recall:



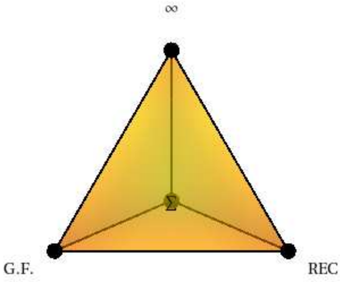
Recall:



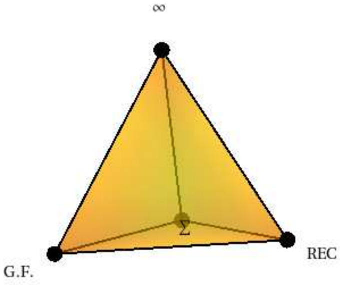
Recall:



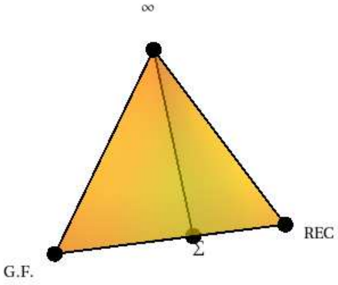
Recall:



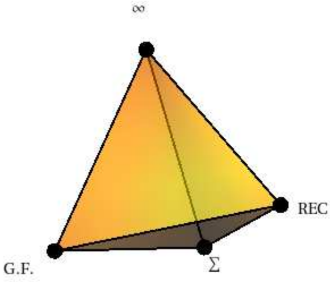
Recall:



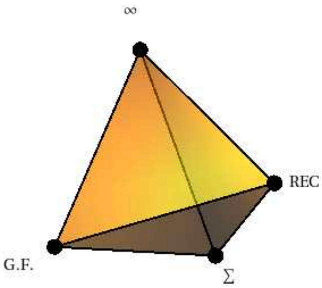
Recall:



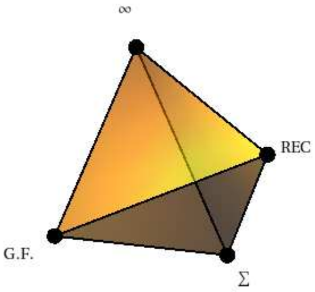
Recall:



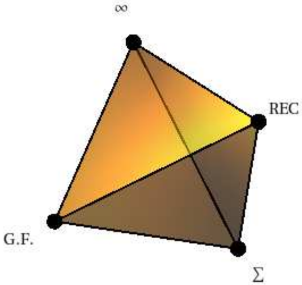
Recall:



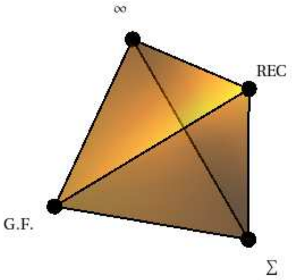
Recall:



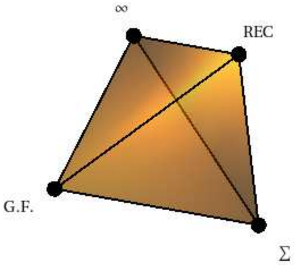
Recall:



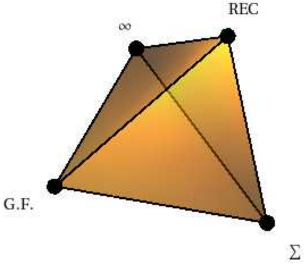
Recall:



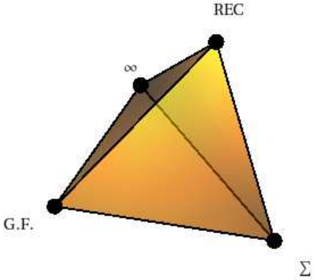
Recall:



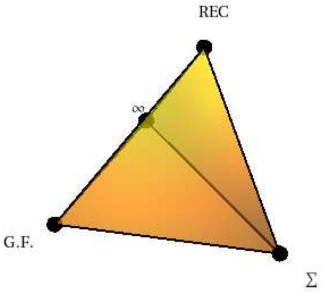
Recall:



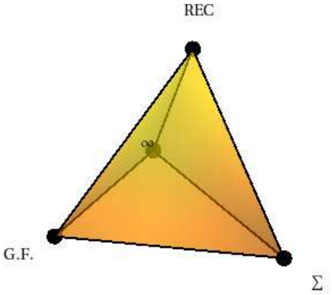
Recall:



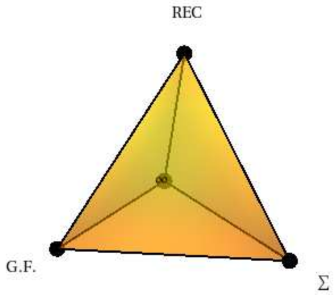
Recall:



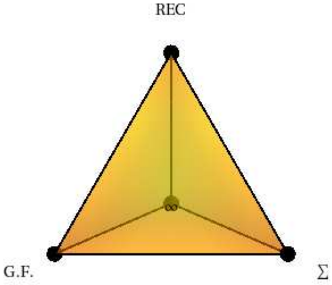
Recall:



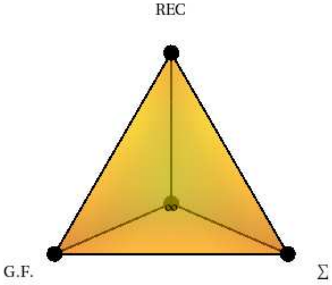
Recall:



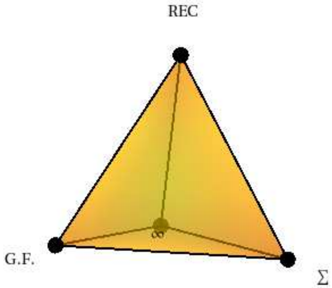
Recall:



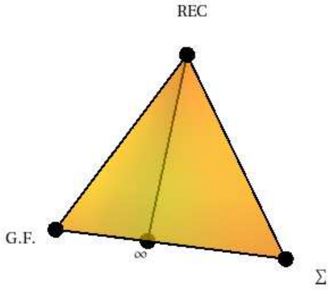
Recall:



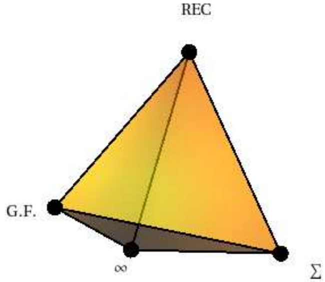
Recall:



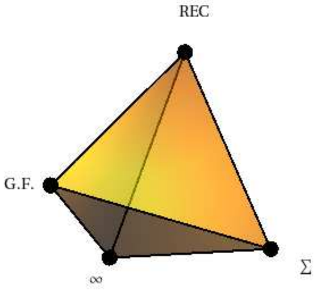
Recall:



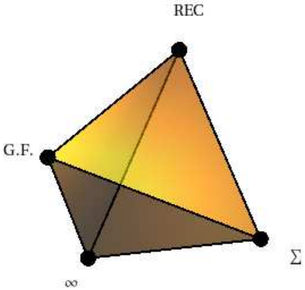
Recall:



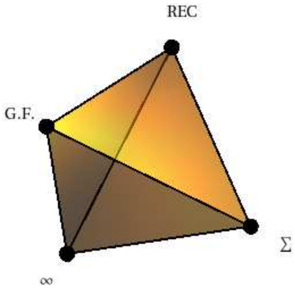
Recall:



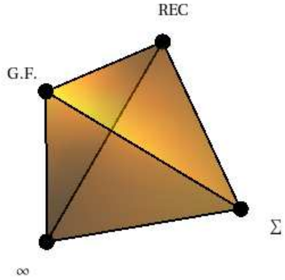
Recall:



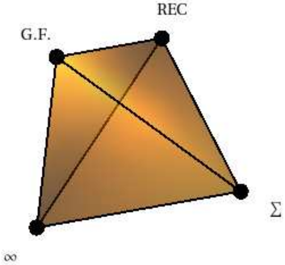
Recall:



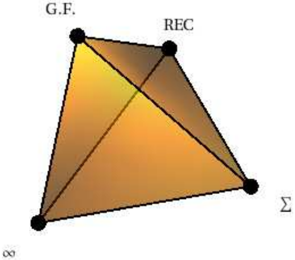
Recall:



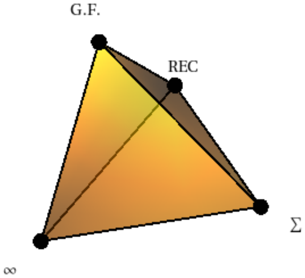
Recall:



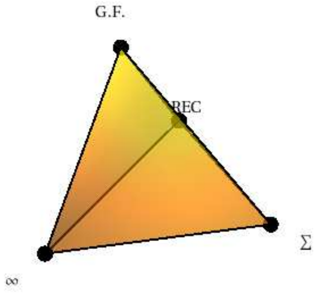
Recall:



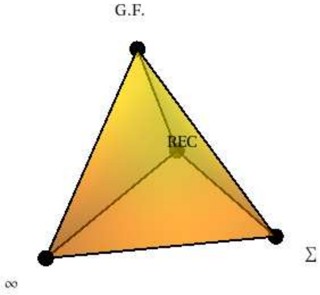
Recall:



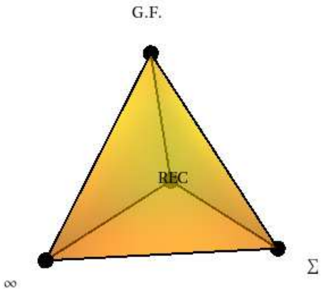
Recall:



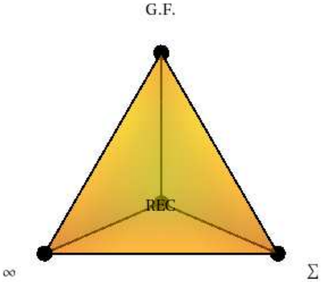
Recall:



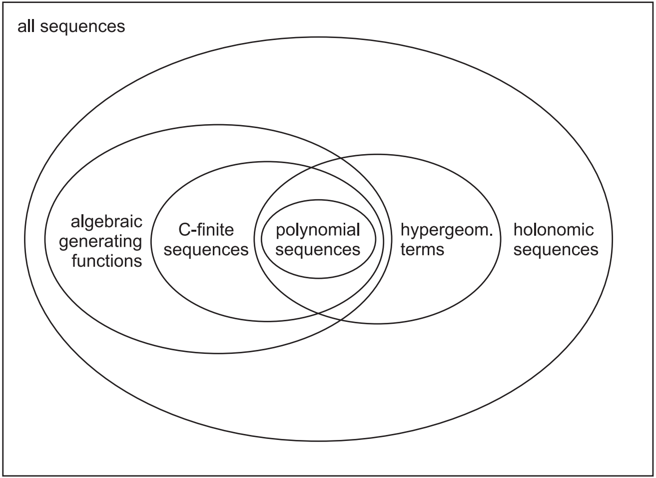
Recall:



Recall:



Recall:


all sequences
algebraic
generating
functions
C-finite
sequences
polynomial
sequences
hypergeom.
terms
holonomic
sequences

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $a_{n+1} - 2a_n = 0$

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $a_{n+1} - 2a_n = 0$
- $n!$:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\quad a_{n+1} - 2a_n = 0$
- $n!$: $\quad a_{n+1} - (n+1)a_n = 0$

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\quad a_{n+1} - 2a_n = 0$
- $n!$: $\quad a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $a_{n+1} - 2a_n = 0$
- $n!$: $a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$: $(n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\quad a_{n+1} - 2a_n = 0$
- $n!$: $\quad a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$: $\quad (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$
- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, ...

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Examples:

- 2^n: $\quad a_{n+1} - 2a_n = 0$
- $n!$: $\quad a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$: $\quad (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$
- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, ...
- Many sequences which have no name and no closed form.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Not holonomic:

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Not holonomic:

- 2^{2^n}.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Not holonomic:

- 2^{2^n}.
- The sequence of prime numbers.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Not holonomic:

- 2^{2^n}.
- The sequence of prime numbers.
- Many sequences which have no name and no closed form.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

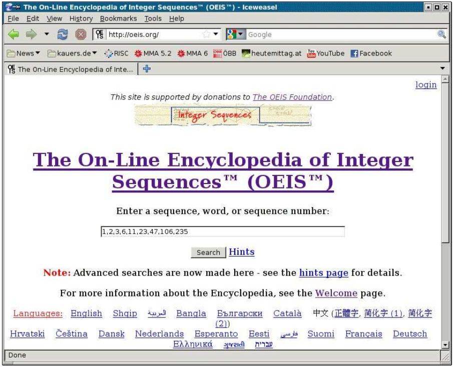
Not holonomic:

- 2^{2^n}.
- The sequence of prime numbers.
- Many sequences which have no name and no closed form.

This means that these sequences can (provably) not be viewed as solutions of a linear recurrence equation with polynomial coefficients.

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$



Approximately 25% of the sequences in Sloane's Online Encyclopedia of Integer Sequences fall into this category.

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

- the recurrence equation

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

- the recurrence equation
- a finite number of initial values $a_0, a_1, a_2, \ldots, a_k$

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

- the recurrence equation
- a finite number of initial values $a_0, a_1, a_2, \ldots, a_k$
  (We can take $k = \max(r, \max\{n \in \mathbb{N} : p_r(n - r) = 0\})$.)

Theorem. The solution set of a linear recurrence equation of order r whose leading coefficient has s integer roots greater than r is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^{\infty}$ is uniquely determined by

- the recurrence equation
- a finite number of initial values $a_0, a_1, a_2, \dots, a_k$
  (We can take $k = \max(r, \max\{n \in \mathbb{N} : p_r(n - r) = 0\})$.)

Consequence: A holonomic sequence can be represented exactly by a finite amount of data.

Examples.

Examples.

- $a_n = 2^n$

Examples.

- $a_n = 2^n \iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$

Examples.

- $a_n = 2^n \qquad \iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n!$

Examples.

- $a_n = 2^n \quad \iff \quad a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n! \quad \iff \quad a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$

Examples.

- $a_n = 2^n \quad \Longleftrightarrow \quad a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n! \quad \Longleftrightarrow \quad a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Examples.

- $a_n = 2^n \iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n! \iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  $\iff (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$
  $a_0 = 1, a_1 = 0$

Examples.

- $a_n = 2^n \iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n! \iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  $\iff (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$
  $a_0 = 1, a_1 = 0$
- $a_n =$ the number of involutions of n letters

Examples.

- $a_n = 2^n \iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n! \iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  $\iff (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$
  $a_0 = 1, a_1 = 0$
- $a_n =$ the number of involutions of n letters
  $\iff a_{n+3} + na_{n+2} - (3n+6)a_{n+1} - (n+1)(n+2)a_n = 0,$
  $a_0 = 1, a_1 = 1, a_2 = 2$

Examples.

- $a_n = 2^n \iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n! \iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  $\iff (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$
  $a_0 = 1, a_1 = 0$
- $a_n =$ the number of involutions of n letters
  $\iff a_{n+3} + na_{n+2} - (3n+6)a_{n+1} - (n+1)(n+2)a_n = 0,$
  $a_0 = 1, a_1 = 1, a_2 = 2$
- $a_n = 0, 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \dots$

Examples.

- $a_n = 2^n \iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n! \iff a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
  $\iff (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0,$
  $a_0 = 1, a_1 = 0$
- $a_n =$ the number of involutions of n letters
  $\iff a_{n+3} + na_{n+2} - (3n+6)a_{n+1} - (n+1)(n+2)a_n = 0,$
  $a_0 = 1, a_1 = 1, a_2 = 2$
- $a_n = 0, 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$
  $\iff (n-6)a_{n+1} - (n-5)a_n = 0,$
  $a_0 = a_1 = \cdots = a_6 = 0, a_7 = 1$

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$:

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\qquad f'(x) - f(x) = 0$

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\qquad f'(x) - f(x) = 0$
- $\log(1-x)$:

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$:

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $f'(x) - f(x) = 0$
- $\log(1-x)$: $(x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$: $(x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$: $\quad (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$
- Bessel functions, Hankel functions, Struve functions, Airy functions, Polylogarithms, Elliptic integrals, the Error function, Kelvin functions, Mathieu functions, ...

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Examples:

- $\exp(x)$: $f'(x) - f(x) = 0$
- $\log(1-x)$: $(x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$: $(x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$
- Bessel functions, Hankel functions, Struve functions, Airy functions, Polylogarithms, Elliptic integrals, the Error function, Kelvin functions, Mathieu functions, ...
- Many functions which have no name and no closed form.

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Not holonomic:

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Not holonomic:

- $\exp(\exp(x) - 1)$.

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Not holonomic:

- $\exp(\exp(x) - 1)$.
- The Riemann Zeta function.

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Not holonomic:

- $\exp(\exp(x) - 1)$.
- The Riemann Zeta function.
- Many functions which have no name and no closed form.

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

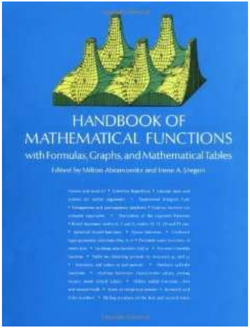
Not holonomic:

- $\exp(\exp(x) - 1)$.
- The Riemann Zeta function.
- Many functions which have no name and no closed form.

This means that these functions can (provably) not be viewed as solutions of a linear differential equation with polynomial coefficients.

Definition ("continuous" case). A formal power series $f \in K[[x]]$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$



Approximately 60% of the functions in Abramowitz and Stegun's handbook fall into this category.

Theorem. A linear differential equation of order r with polynomial coefficients can have at most r linearly independent solutions in $K[[x]]$.

Theorem. A linear differential equation of order r with polynomial coefficients can have at most r linearly independent solutions in $K[[x]]$.

Consequence: A holonomic power series is uniquely determined by

Theorem. A linear differential equation of order r with polynomial coefficients can have at most r linearly independent solutions in $K[[x]]$.

Consequence: A holonomic power series is uniquely determined by

- the differential equation

Theorem. A linear differential equation of order r with polynomial coefficients can have at most r linearly independent solutions in $K[[x]]$.

Consequence: A holonomic power series is uniquely determined by

- the differential equation
- a finite number of initial terms $f(0), f'(0), f''(0), \ldots, f^{(k)}(0)$

Theorem. A linear differential equation of order r with polynomial coefficients can have at most r linearly independent solutions in $K[[x]]$.

Consequence: A holonomic power series is uniquely determined by

- the differential equation
- a finite number of initial terms $f(0), f'(0), f''(0), \dots, f^{(k)}(0)$ (Usually, $k = r$ suffices.)

Theorem. A linear differential equation of order r with polynomial coefficients can have at most r linearly independent solutions in $K[[x]]$.

Consequence: A holonomic power series is uniquely determined by

- the differential equation
- a finite number of initial terms $f(0), f'(0), f''(0), \dots, f^{(k)}(0)$ (Usually, $k = r$ suffices.)

Consequence: A holonomic power series can be represented exactly by a finite amount of data.

Examples.

Examples.

- $f(x) = \exp(x)$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$
  $\iff (x - 1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$
  $\iff (x - 1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$
- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$
  $\iff (x - 1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$
- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$
  $\iff (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0,$
  $f(0) = \frac{1}{2}, f'(0) = 0$

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1 - x)$
  $\iff (x-1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$
- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$
  $\iff (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0,$
  $f(0) = \frac{1}{2}, f'(0) = 0$
- $f(x) =$ the fifth modified Bessel function of the first kind

Examples.

- $f(x) = \exp(x)$
  $\iff f'(x) - f(x) = 0, \quad f(0) = 1$
- $f(x) = \log(1-x)$
  $\iff (x-1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1$
- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$
  $\iff (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0,$
  $f(0) = \frac{1}{2}, f'(0) = 0$
- $f(x) =$ the fifth modified Bessel function of the first kind
  $\iff x^2 f''(x) + xf'(x) - (x^2 + 25)f(x) = 0,$
  $f(0) = f'(0) = \cdots = f^{(4)}(0) = 0, f^{(5)}(0) = \frac{1}{32}$

$1, \quad 2, \quad 14, \quad 106, \quad 838, \quad 6802, \quad 56190, \quad 470010, \quad 3968310,$
$33747490, \quad 288654574, \quad 2480593546, \quad 21400729382,$
$185239360178, \quad 1607913963614, \quad 13991107041306,$
$122002082809110, \quad 1065855419418690, \quad 9327252391907790$
$81744134786314410, \quad 9327252391907790, \quad \dots$

$$1,\quad 2,\quad 14,\quad 106,\quad 838,\quad 6802,\quad 56190,\quad 470010,\quad 3968310,$$
$$33747490,\quad 288654574,\quad 2480593546,\quad 21400729382,$$
$$185239360178,\quad 1607913963614,\quad 13991107041306,$$
$$122002082809110,\quad 1065855419418690,\quad 9327252391907790$$
$$81744134786314410,\quad 9327252391907790,\quad \dots$$

Is this a holonomic sequence?

Let's see whether the data satisfies a recurrence of the form

$$(c_{0,0}+c_{0,1}n)a_{n,n}+(c_{1,0}+c_{1,1}n)a_{n+1,n+1}+(c_{2,0}+c_{2,1}n)a_{n+2,n+2}=0$$

where the $c_{i,j}$ are some as yet unknown numbers.

Let's see whether the data satisfies a recurrence of the form

$$(c_{0,0}+c_{0,1}n)a_{n,n}+(c_{1,0}+c_{1,1}n)a_{n+1,n+1}+(c_{2,0}+c_{2,1}n)a_{n+2,n+2}=0$$

where the $c_{i,j}$ are some as yet unknown numbers.

If we won't find any recurrence of this form, we can try again with higher order and/or higher degree.

Match the recurrence template (“ansatz”) against the data.

Match the recurrence template ("ansatz") against the data.

$$n = 0: \ (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0$$

Match the recurrence template ("ansatz") against the data.

$$n = 0: \ (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0$$
$$n = 1: \ (c_{0,0} + c_{0,1}1)2 + (c_{1,0} + c_{1,1}1)14 + (c_{2,0} + c_{2,1}1)106 = 0$$

Match the recurrence template ("ansatz") against the data.

$$
\begin{aligned}
n = 0: &\ (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0 \\
n = 1: &\ (c_{0,0} + c_{0,1}1)2 + (c_{1,0} + c_{1,1}1)14 + (c_{2,0} + c_{2,1}1)106 = 0 \\
n = 2: &\ (c_{0,0} + c_{0,1}2)14 + (c_{1,0} + c_{1,1}2)106 + (c_{2,0} + c_{2,1}2)838 = 0
\end{aligned}
$$

Match the recurrence template ("ansatz") against the data.

$$
\begin{aligned}
n = 0:&\ \ (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0\\
n = 1:&\ \ (c_{0,0} + c_{0,1}1)2 + (c_{1,0} + c_{1,1}1)14 + (c_{2,0} + c_{2,1}1)106 = 0\\
n = 2:&\ \ (c_{0,0} + c_{0,1}2)14 + (c_{1,0} + c_{1,1}2)106 + (c_{2,0} + c_{2,1}2)838 = 0\\
&\ \ \vdots\\
n = 8:&\ \ (c_{0,0} + c_{0,1}8)3968310 + (c_{1,0} + c_{1,1}8)33747490\\
&\qquad + (c_{2,0} + c_{2,1}8)288654574 = 0
\end{aligned}
$$

Match the recurrence template ("ansatz") against the data.

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 14 & 0 \\ 2 & 2 & 14 & 14 & 106 & 106 \\ 14 & 28 & 106 & 212 & 838 & 1676 \\ 106 & 318 & 838 & 2514 & 6802 & 20406 \\ 838 & 3352 & 6802 & 27208 & 56190 & 224760 \\ 6802 & 34010 & 56190 & 280950 & 470010 & 2350050 \\ 56190 & 337140 & 470010 & 2820060 & 3968310 & 23809860 \\ 470010 & 3290070 & 3968310 & 27778170 & 33747490 & 236232430 \\ 3968310 & 31746480 & 33747490 & 269979920 & 288654574 & 2309236592 \end{pmatrix} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{1,0} \\ c_{1,1} \\ c_{2,0} \\ c_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Match the recurrence template ("ansatz") against the data.

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 14 & 0 \\ 2 & 2 & 14 & 14 & 106 & 106 \\ 14 & 28 & 106 & 212 & 838 & 1676 \\ 106 & 318 & 838 & 2514 & 6802 & 20406 \\ 838 & 3352 & 6802 & 27208 & 56190 & 224760 \\ 6802 & 34010 & 56190 & 280950 & 470010 & 2350050 \\ 56190 & 337140 & 470010 & 2820060 & 3968310 & 23809860 \\ 470010 & 3290070 & 3968310 & 27778170 & 33747490 & 236232430 \\ 3968310 & 31746480 & 33747490 & 269979920 & 288654574 & 2309236592 \end{pmatrix} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{1,0} \\ c_{1,1} \\ c_{2,0} \\ c_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve this linear system!

Match the recurrence template ("ansatz") against the data.

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 14 & 0 \\ 2 & 2 & 14 & 14 & 106 & 106 \\ 14 & 28 & 106 & 212 & 838 & 1676 \\ 106 & 318 & 838 & 2514 & 6802 & 20406 \\ 838 & 3352 & 6802 & 27208 & 56190 & 224760 \\ 6802 & 34010 & 56190 & 280950 & 470010 & 2350050 \\ 56190 & 337140 & 470010 & 2820060 & 3968310 & 23809860 \\ 470010 & 3290070 & 3968310 & 27778170 & 33747490 & 236232430 \\ 3968310 & 31746480 & 33747490 & 269979920 & 288654574 & 2309236592 \end{pmatrix} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{1,0} \\ c_{1,1} \\ c_{2,0} \\ c_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve this linear system!

Since there are more equations than variables, we expect 0 solutions.

Strangely enough, there happens to be a solution!

$$(c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}) = (0, 9, -14, -10, 2, 1)$$

Strangely enough, there happens to be a solution!

$$(c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}) = (0, 9, -14, -10, 2, 1)$$

It follows that for $n = 0, 1, 2, \ldots, 8$ we have

$$9n\, a_n - (10n + 14)a_{n+1} + (n + 2)a_{n+2} = 0$$

Strangely enough, there happens to be a solution!

$$(c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}) = (0, 9, -14, -10, 2, 1)$$

It follows that for $n = 0, 1, 2, \ldots, 8$ we have

$$9n\, a_n - (10n + 14)a_{n+1} + (n + 2)a_{n+2} = 0$$

Even more strangely, this recurrence continues to hold for $n = 9, 10, \ldots, 15$, even though these terms were not used during the computation.

Strangely enough, there happens to be a solution!

$$(c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}) = (0, 9, -14, -10, 2, 1)$$

It follows that for $n = 0, 1, 2, \ldots, 8$ we have

$$9n\, a_n - (10n + 14)a_{n+1} + (n + 2)a_{n+2} = 0$$

Even more strangely, this recurrence continues to hold for $n = 9, 10, \ldots, 15$, even though these terms were not used during the computation.

Either we witness a **veeeery** unlikely coincidence,
or we have indeed found a recurrence which has some meaning.

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

It's pretty the same as for algebraic numbers.

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of “closed form”.

It’s pretty the same as for algebraic numbers.

Naive question: What are the roots of the polynomial $x^5 - 3x + 1$?

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

It's pretty the same as for algebraic numbers.

Naive question: What are the roots of the polynomial $x^5 - 3x + 1$?

Expert answer:
$$\begin{aligned}&\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 1),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 2),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 3),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 4),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 5).\end{aligned}$$

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

For holonomic sequences:

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

For holonomic sequences:

Naive question: What are the solutions of the recurrence

$$(3n+2)a_{n+2} - 2(n+3)a_{n+1} + (2n-7)a_n = 0 \ ?$$

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

For holonomic sequences:

Naive question: What are the solutions of the recurrence

$$(3n+2)a_{n+2} - 2(n+3)a_{n+1} + (2n-7)a_n = 0 \ ?$$

A holonomist's answer: There is exactly one solution with $a_0 = 0$, $a_1 = 1$, exactly one solution with $a_0 = 1$, $a_1 = 0$, and every other solution is a K-linear combination of those two.

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

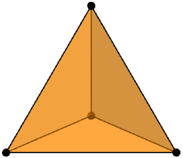
When computing with holonomic objects, we compute with the equations through which they are defined.

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

When computing with holonomic objects, we compute with the equations through which they are defined.

Like before, our goal is to establish computational links between

- recurrence equations
- generating functions
- asymptotic estimates
- symbolic sums



A Recurrence equations:

A Recurrence equations:

Trivial: Holonomic sequences are *given* in terms of a recurrence.

B Generating Functions

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$
\begin{aligned}
& (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\
\iff \quad & a(x) \text{ is holonomic as a power series}
\end{aligned}
$$

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\ \iff \quad & a(x) \text{ is holonomic as a power series} \end{aligned}$$

The theorem is algorithmic:

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\ \iff \quad & a(x) \text{ is holonomic as a power series} \end{aligned}$$

The theorem is algorithmic:

- Given a recurrence for $(a_n)_{n=0}^{\infty}$, we can compute a differential equation for $a(x)$.

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\ \iff \quad & a(x) \text{ is holonomic as a power series} \end{aligned}$$

The theorem is algorithmic:

- Given a recurrence for $(a_n)_{n=0}^{\infty}$, we can compute a differential equation for $a(x)$.
- Given a differential equation for $a(x)$, we can compute a recurrence for $(a_n)_{n=0}^{\infty}$.

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\ \iff \quad & a(x) \text{ is holonomic as a power series} \end{aligned}$$

Examples.

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$
\begin{aligned}
& (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\
\iff \quad & a(x) \text{ is holonomic as a power series}
\end{aligned}
$$

Examples.

INPUT: $a'(x) - a(x) = 0, a(0) = 1$ (i.e., $a(x) = \exp(x)$)

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\ \iff \quad & a(x) \text{ is holonomic as a power series} \end{aligned}$$

Examples.

INPUT: $a'(x) - a(x) = 0, a(0) = 1$ (i.e., $a(x) = \exp(x)$)

⬇

OUTPUT: $(n+1)a_{n+1} - a_n = 0, a_0 = 1$ (i.e., $a_n = \frac{1}{n!}$)

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\ \iff \quad & a(x) \text{ is holonomic as a power series} \end{aligned}$$

Examples.

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\ \iff \quad & a(x) \text{ is holonomic as a power series} \end{aligned}$$

Examples.

INPUT: $2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0$

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{array}{cl} & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\ \iff & a(x) \text{ is holonomic as a power series} \end{array}$$

Examples.

INPUT: $2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0$

⬇

OUTPUT: $x^5 a^{(5)}(x) + (19x^2 + 3x - 1)x^2 a^{(4)}(x)$
$+ 2(55x^3 + 15x^2 - 2x - 1)a^{(3)}(x) + 6(37x + 12)xa''(x)$
$+ 12(11x + 3)a'(x) + 12a(x) = 0$

C Asymptotic Estimates

C Asymptotic Estimates

Theorem.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^\alpha \log(n)^\beta \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^\alpha \log(n)^\beta \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\, \mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.
- If $a(x)$ is holonomic and, as an analytic function, has a singularity at ζ, then

$$a(x) \sim c\,\mathrm{e}^{P((\zeta - x)^{-1/r})} (\zeta - x)^{\alpha} \log(\zeta - x)^{\beta} \quad (x \to \zeta)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, α is a constant, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

- If $a(x)$ is holonomic and, as an analytic function, has a singularity at ζ, then

$$a(x) \sim c\,\mathrm{e}^{P((\zeta - x)^{-1/r})} (\zeta - x)^{\alpha} \log(\zeta - x)^{\beta} \quad (x \to \zeta)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, α is a constant, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\, \mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

- If $a(x)$ is holonomic and, as an analytic function, has a singularity at ζ, then

$$a(x) \sim c\, \mathrm{e}^{P((\zeta - x)^{-1/r})} (\zeta - x)^{\alpha} \log(\zeta - x)^{\beta} \quad (x \to \zeta)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, α is a constant, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

- If $a(x)$ is holonomic and, as an analytic function, has a singularity at ζ, then

$$a(x) \sim c\,\mathrm{e}^{P((\zeta - x)^{-1/r})} (\zeta - x)^{\alpha} \log(\zeta - x)^{\beta} \quad (x \to \zeta)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, α is a constant, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\,\mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

- If $a(x)$ is holonomic and, as an analytic function, has a singularity at ζ, then

$$a(x) \sim c\,\mathrm{e}^{P((\zeta - x)^{-1/r})} (\zeta - x)^{\alpha} \log(\zeta - x)^{\beta} \quad (x \to \zeta)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, α is a constant, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

Theorem.

- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

$$a_n \sim c\, \mathrm{e}^{P(n^{1/r})} n^{\gamma n} \phi^n n^{\alpha} \log(n)^{\beta} \quad (n \to \infty)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, γ, ϕ, α are constants, and $\beta \in \mathbb{N}$.

- If $a(x)$ is holonomic and, as an analytic function, has a singularity at ζ, then

$$a(x) \sim c\, \mathrm{e}^{P((\zeta - x)^{-1/r})} (\zeta - x)^{\alpha} \log(\zeta - x)^{\beta} \quad (x \to \zeta)$$

  where c is a constant, P is a polynomial, $r \in \mathbb{N}$, α is a constant, and $\beta \in \mathbb{N}$.

C Asymptotic Estimates

C Asymptotic Estimates

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.

C Asymptotic Estimates

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.

C Asymptotic Estimates

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

C Asymptotic Estimates

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

Example.

C Asymptotic Estimates

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

Example.
INPUT:
$2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0, a_0 = a_1 = 1$

C Asymptotic Estimates

- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

Example.

INPUT:

$$2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0, a_0 = a_1 = 1$$

⬇

OUTPUT:

$$c\,\mathrm{e}^{\sqrt{n}-\frac{n}{2}} n^{n/2} \left(1 - \tfrac{119}{1152} n^{-1} + \tfrac{7}{24} n^{-1/2} + \tfrac{1967381}{39813120} n^{-2} + \mathrm{O}(n^{-3/2})\right)$$

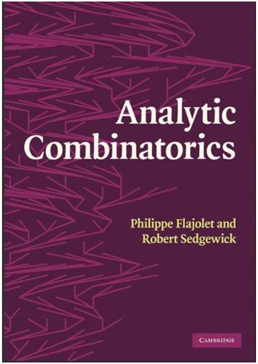
with $c \approx 0.5506953149031837476159810627496478467138 2\ldots$

C Asymptotic Estimates

An excellent reference for modern techniques for computing asymptotic estimates is:

C Asymptotic Estimates

An excellent reference for modern techniques for computing asymptotic estimates is:



D Symbolic Summation

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Proof:

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Proof:

- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum\limits_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Proof:

- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.
- $(a_n)_{n=0}^{\infty}$ is holonomic by assumption.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Proof:

- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.
- $(a_n)_{n=0}^{\infty}$ is holonomic by assumption.
- Therefore $a(x)$ is holonomic as power series.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Proof:

- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.
- $(a_n)_{n=0}^{\infty}$ is holonomic by assumption.
- Therefore $a(x)$ is holonomic as power series.
- This means $a(x)$ satisfies a differential equation.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Proof:

- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.
- $(a_n)_{n=0}^{\infty}$ is holonomic by assumption.
- Therefore $a(x)$ is holonomic as power series.
- This means $a(x)$ satisfies a differential equation.
- Apply the substitution $a(x) = (1-x)b(x)$.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Proof:

- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.
- $(a_n)_{n=0}^{\infty}$ is holonomic by assumption.
- Therefore $a(x)$ is holonomic as power series.
- This means $a(x)$ satisfies a differential equation.
- Apply the substitution $a(x) = (1 - x)b(x)$.
- It follows that $b(x)$ satisfies a differential equation.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Proof:

- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.
- $(a_n)_{n=0}^{\infty}$ is holonomic by assumption.
- Therefore $a(x)$ is holonomic as power series.
- This means $a(x)$ satisfies a differential equation.
- Apply the substitution $a(x) = (1 - x)b(x)$.
- It follows that $b(x)$ satisfies a differential equation.
- This means $b(x)$ is holonomic.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Proof:

- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.
- $(a_n)_{n=0}^{\infty}$ is holonomic by assumption.
- Therefore $a(x)$ is holonomic as power series.
- This means $a(x)$ satisfies a differential equation.
- Apply the substitution $a(x) = (1-x)b(x)$.
- It follows that $b(x)$ satisfies a differential equation.
- This means $b(x)$ is holonomic.
- Therefore $(b_n)_{n=0}^{\infty}$ is holonomic. ■

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Example:

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Example:

- $a_{n+3} + n a_{n+2} - (3n+6) a_{n+1} - (n+1)(n+2) a_n = 0$

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Example:

- $a_{n+3} + na_{n+2} - (3n+6)a_{n+1} - (n+1)(n+2)a_n = 0$
- $\Longrightarrow (x+1)(2x-1)x^5 a^{(3)}(x) + (\dots)a''(x) + (\dots)a'(x) + (4x^4 + 4x^3 - 7x^2 - 2x - 1)a(x) = 0$

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum\limits_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Example:

- $a_{n+3} + na_{n+2} - (3n+6)a_{n+1} - (n+1)(n+2)a_n = 0$
- $\implies (x+1)(2x-1)x^5 a^{(3)}(x) + (\dots)a''(x) + (\dots)a'(x) + (4x^4 + 4x^3 - 7x^2 - 2x - 1)a(x) = 0$
- $\implies (x-1)(x+1)(2x-1)x^5 b^{(3)}(x) + (\dots)b''(x) + (\dots)b'(x) + 2(12x^5 + 13x^4 - 8x^3 - 4x^2 + 1)b(x) = 0$

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Example:

- $a_{n+3} + n a_{n+2} - (3n+6) a_{n+1} - (n+1)(n+2) a_n = 0$
- $\implies (x+1)(2x-1)x^5 a^{(3)}(x) + (\dots)a''(x) + (\dots)a'(x) + (4x^4 + 4x^3 - 7x^2 - 2x - 1)a(x) = 0$
- $\implies (x-1)(x+1)(2x-1)x^5 b^{(3)}(x) + (\dots)b''(x) + (\dots)b'(x) + 2(12x^5 + 13x^4 - 8x^3 - 4x^2 + 1)b(x) = 0$
- $\implies 2(n+3)(n+2)^2 b_n - (n+3)(n^2 - 6n - 20)b_{n+1} - (n+10)(2n^2 + 11n + 16)b_{n+2} + (n-1)(n^2 + 11n + 26)b_{n+3} + (n+4)(5n+29)b_{n+4} - (n^2 + 7n + 8)b_{n+5} - (n+6)b_{n+6} = 0$

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Remarks:

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Remarks:

- This is not the algorithm of choice.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Remarks:

- This is not the algorithm of choice.
- With a less brutal algorithm one can find for every sum a recurrence whose order is at most one more than the order of the recurrence of the summand.

D Symbolic Summation

If $(a_n)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^{\infty}$ is holonomic.

Remarks:

- This is not the algorithm of choice.
- With a less brutal algorithm one can find for every sum a recurrence whose order is at most one more than the order of the recurrence of the summand.
- There is also an algorithm due to Abramov and van Hoeij for computing "closed form" solutions of holonomic sums in terms of the summand, such as

$$\sum_{k=0}^{n}\Big(\frac{2k+5}{k+2}F_k - \frac{k+4}{k+3}F_{k+1}\Big) = F_n - \frac{1}{n+3}F_{n+1} - 1.$$

Closure properties:

Closure properties:

We have just seen: summation preserves holonomy.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.
- if $u, v \in \mathbb{Q}$ are positive, then $(a_{\lfloor un+v \rfloor})_{n=0}^{\infty}$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.
- if $u, v \in \mathbb{Q}$ are positive, then $(a_{\lfloor un+v \rfloor})_{n=0}^{\infty}$ is holonomic.

Recurrence equations for all these sequences can be computed from given defining equations of $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $a(x)$ and $b(x)$ be holonomic power series. Then:

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.
- if $b(x)$ is **algebraic** and $b(0) = 0$, then $a(b(x))$ is holonomic.

Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.
- if $b(x)$ is **algebraic** and $b(0) = 0$, then $a(b(x))$ is holonomic.

Differential equations for all these functions can be computed from given defining equations of $a(x)$ and $b(x)$.

Closure properties:

Closure properties: Why true?

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$a(x)$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$a(x), a'(x)$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$a(x), a'(x), a''(x)$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$a(x), a'(x), a''(x), a'''(x)$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$a(x), a'(x), a''(x), a'''(x), \ldots$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)}\text{-VS}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim \langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim \langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim \langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$a(x)b(x),$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim \langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$a(x)b(x), \quad a'(x)b(x),$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim\langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$a(x)b(x), \quad a'(x)b(x), \quad a''(x)b(x),$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-vs}} < \infty$$
$$\dim \langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-vs}} < \infty.$$

For the same reason,

$$a(x)b(x), \quad a'(x)b(x), \quad a''(x)b(x), \quad \dots$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim \langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$\begin{array}{llll} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & & & \end{array}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim \langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$\begin{array}{llll} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & & \end{array}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim\langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$\begin{array}{llll} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & a''(x)b'(x), & \end{array}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim\langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$\begin{array}{llll} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & a''(x)b'(x), & \dots \end{array}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim\langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$\begin{matrix} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & a''(x)b'(x), & \dots \\ a(x)b''(x), & a'(x)b''(x), & a''(x)b''(x), & \dots \end{matrix}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim \langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$\begin{matrix} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & a''(x)b'(x), & \dots \\ a(x)b''(x), & a'(x)b''(x), & a''(x)b''(x), & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim \langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$\Big\langle \begin{matrix} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & a''(x)b'(x), & \dots \\ a(x)b''(x), & a'(x)b''(x), & a''(x)b''(x), & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \Big\rangle_{K(x)\text{-VS}}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim\langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$V := \Big\langle \begin{matrix} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & a''(x)b'(x), & \dots \\ a(x)b''(x), & a'(x)b''(x), & a''(x)b''(x), & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \Big\rangle_{K(x)\text{-VS}}$$

Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim\langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$V := \Big\langle \begin{matrix} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & a''(x)b'(x), & \dots \\ a(x)b''(x), & a'(x)b''(x), & a''(x)b''(x), & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \Big\rangle_{K(x)\text{-VS}}$$

has a **finite dimension.**

Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$.

Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$. Then:

Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$. Then:

$$c'(x) = a'(x)b(x) + a(x)b'(x)$$

Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$. Then:

$$c'(x) = a'(x)b(x) + a(x)b'(x)$$
$$c''(x) = a''(x)b(x) + 2a'(x)b'(x) + a(x)b''(x)$$

Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$. Then:

$$\begin{aligned} c'(x) &= a'(x)b(x) + a(x)b'(x) \\ c''(x) &= a''(x)b(x) + 2a'(x)b'(x) + a(x)b''(x) \\ c'''(x) &= a'''(x)b(x) + 3a''(x)b'(x) + 3a'(x)b''(x) + a(x)b'''(x) \end{aligned}$$

Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$. Then:

$$\begin{aligned}
c'(x) &= a'(x)b(x) + a(x)b'(x) \\
c''(x) &= a''(x)b(x) + 2a'(x)b'(x) + a(x)b''(x) \\
c'''(x) &= a'''(x)b(x) + 3a''(x)b'(x) + 3a'(x)b''(x) + a(x)b'''(x) \\
&\vdots
\end{aligned}$$

Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$. Then:

$$\begin{aligned} c'(x) &= a'(x)b(x) + a(x)b'(x) \\ c''(x) &= a''(x)b(x) + 2a'(x)b'(x) + a(x)b''(x) \\ c'''(x) &= a'''(x)b(x) + 3a''(x)b'(x) + 3a'(x)b''(x) + a(x)b'''(x) \\ &\vdots \end{aligned}$$

This means all the $c^{(i)}(x)$ belong to the vector space V.

Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$. Then:

$$\begin{aligned} c'(x) &= a'(x)b(x) + a(x)b'(x) \\ c''(x) &= a''(x)b(x) + 2a'(x)b'(x) + a(x)b''(x) \\ c'''(x) &= a'''(x)b(x) + 3a''(x)b'(x) + 3a'(x)b''(x) + a(x)b'''(x) \\ &\vdots \end{aligned}$$

This means all the $c^{(i)}(x)$ belong to the vector space V.

Therefore, $c(x), c'(x), c''(x), \ldots, c^{(r)}(x)$ must be linearly dependent over $K(x)$ as soon as $r > \dim V$.

Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$. Then:

$$\begin{aligned} c'(x) &= a'(x)b(x) + a(x)b'(x) \\ c''(x) &= a''(x)b(x) + 2a'(x)b'(x) + a(x)b''(x) \\ c'''(x) &= a'''(x)b(x) + 3a''(x)b'(x) + 3a'(x)b''(x) + a(x)b'''(x) \\ &\vdots \end{aligned}$$

This means all the $c^{(i)}(x)$ belong to the vector space V.

Therefore, $c(x), c'(x), c''(x), \dots, c^{(r)}(x)$ must be linearly dependent over $K(x)$ as soon as $r > \dim V$.

In other words, $c(x)$ must be holonomic.

Closure properties: Why true?

The other closure properties are proved by similar arguments.

Closure properties: Why algorithmic?

Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:

Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:

- Make an ansatz $p_0(x)c(x) + p_1(x)c'(x) + \cdots + p_r(x)c^{(r)}(x)$ with undetermined coefficients $p_k(x)$.

Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:

- Make an ansatz $p_0(x)c(x) + p_1(x)c'(x) + \cdots + p_r(x)c^{(r)}(x)$ with undetermined coefficients $p_k(x)$.
- Use the defining equations of $a(x)$ and $b(x)$ to rewrite the higher order derivatives in $c^{(k)}(x) = D_x^k\bigl(a(x)b(x)\bigr)$ in terms of lower order ones.

Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:

- Make an ansatz $p_0(x)c(x) + p_1(x)c'(x) + \cdots + p_r(x)c^{(r)}(x)$ with undetermined coefficients $p_k(x)$.
- Use the defining equations of $a(x)$ and $b(x)$ to rewrite the higher order derivatives in $c^{(k)}(x) = D_x^k\big(a(x)b(x)\big)$ in terms of lower order ones.
- Compare coefficients of $a^{(i)}(x)b^{(j)}(x)$ to zero.

Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:

- Make an ansatz $p_0(x)c(x) + p_1(x)c'(x) + \cdots + p_r(x)c^{(r)}(x)$ with undetermined coefficients $p_k(x)$.
- Use the defining equations of $a(x)$ and $b(x)$ to rewrite the higher order derivatives in $c^{(k)}(x) = D_x^k\big(a(x)b(x)\big)$ in terms of lower order ones.
- Compare coefficients of $a^{(i)}(x)b^{(j)}(x)$ to zero.
- This gives a linear system over $K(x)$ for the coefficients $p_k(x)$ which will have a solution if r is big enough.

Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:

- Make an ansatz $p_0(x)c(x) + p_1(x)c'(x) + \cdots + p_r(x)c^{(r)}(x)$ with undetermined coefficients $p_k(x)$.
- Use the defining equations of $a(x)$ and $b(x)$ to rewrite the higher order derivatives in $c^{(k)}(x) = D_x^k\big(a(x)b(x)\big)$ in terms of lower order ones.
- Compare coefficients of $a^{(i)}(x)b^{(j)}(x)$ to zero.
- This gives a linear system over $K(x)$ for the coefficients $p_k(x)$ which will have a solution if r is big enough.

Packages like gfun (for Maple) or GeneratingFunctions.m (for Mathematica) do this for you.

Closure properties: Why interesting?

Closure properties: Why interesting?

Algorithms for "executing closure properties" are useful for proving identities among holonomic sequences and power series.

Closure properties: Why interesting?

Algorithms for "executing closure properties" are useful for proving identities among holonomic sequences and power series.

Basic idea: $A = B \iff A - B = 0$

Closure properties: Why interesting?

Algorithms for "executing closure properties" are useful for proving identities among holonomic sequences and power series.

Basic idea: $A = B \iff A - B = 0$

Once we have a recurrence equation for $A - B$, we can prove by induction that it is identically zero.

Closure properties: Why interesting?

Algorithms for "executing closure properties" are useful for proving identities among holonomic sequences and power series.

Basic idea: $A = B \iff A - B = 0$

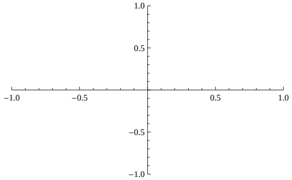
Once we have a recurrence equation for $A - B$, we can prove by induction that it is identically zero.

Let's see two examples.

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

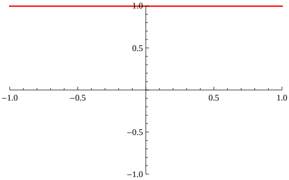
Legendre polynomials:



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

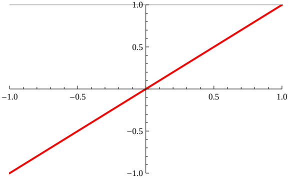
- $P_0(x) = 1$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

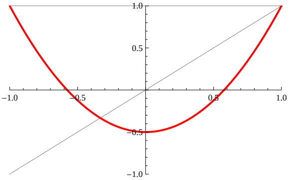
- $P_0(x) = 1$
- $P_1(x) = x$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

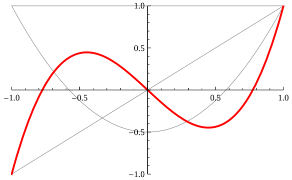
- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

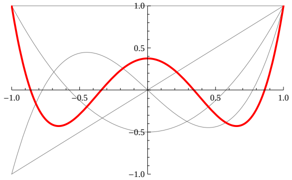
- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

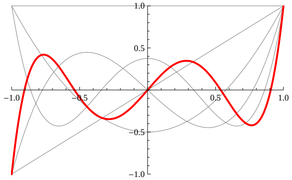
- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
- $P_5(x) = \frac{1}{8}(15x - 70x^3 + 63x^5)$
- $\dots$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

$$P_{n+2}(x) = -\frac{n+1}{n+2}P_n(x) + \frac{2n+3}{n+2}xP_{n+1}(x)$$

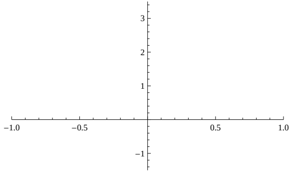
$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

$$\begin{aligned} P_{n+2}(x) &= -\frac{n+1}{n+2}P_n(x) + \frac{2n+3}{n+2}xP_{n+1}(x) \\ P_0(x) &= 1 \\ P_1(x) &= x \end{aligned}$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

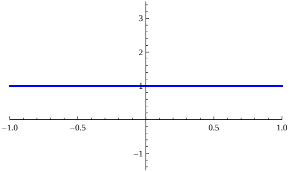
Jacobi polynomials:



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

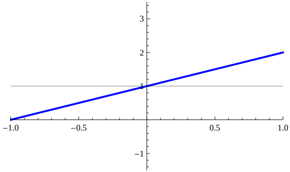
- $P_0^{(1,-1)}(x) = 1$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

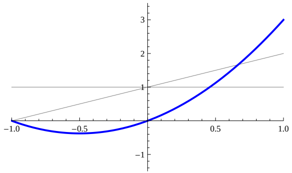
- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

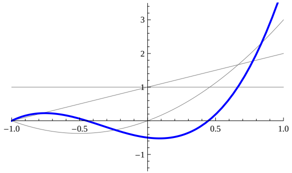
Jacobi polynomials:

- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

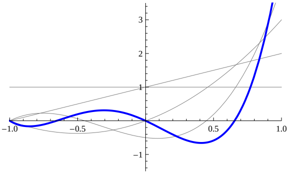
Jacobi polynomials:



- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$
- $P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1} P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

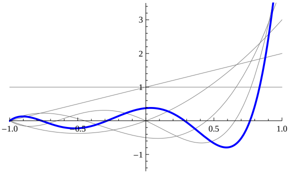
Jacobi polynomials:



- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$
- $P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)$
- $P_4^{(1,-1)}(x) = \frac{5}{8}(-3x - 3x^2 + 7x^3 + 7x^4)$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:



- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$
- $P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)$
- $P_4^{(1,-1)}(x) = \frac{5}{8}(-3x - 3x^2 + 7x^3 + 7x^4)$
- $P_5^{(1,-1)}(x) = \frac{3}{8}(1 + x - 14x^2 - 14x^3 + 21x^4 + 21x^5)$
- $\dots$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

$$P_{n+2}^{(1,-1)}(x) = -\frac{n}{n+1} P_n^{(1,-1)}(x) + \frac{2n+3}{n+2} x P_{n+1}^{(1,-1)}(x)$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

$$P_{n+2}^{(1,-1)}(x) = -\frac{n}{n+1}P_n^{(1,-1)}(x) + \frac{2n+3}{n+2}xP_{n+1}^{(1,-1)}(x)$$
$$P_0^{(1,-1)}(x) = 1$$
$$P_1^{(1,-1)}(x) = 1 + x$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

How to prove this identity?

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

How to prove this identity? $\longrightarrow$ By induction!

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

How to prove this identity? $\longrightarrow$ By induction!

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

How to prove this identity? $\longrightarrow$ By induction!

Compute a recurrence for the left hand side from the defining equations of its building blocks.

$$\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\\text{of order 1}}} \; \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\\text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - \underbrace{P_n(x)}_{\text{recurrence of order 2}} - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - \underbrace{P_n(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - \underbrace{P_{n+1}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Bigl(2 - \underbrace{\underbrace{P_n(x)}_{\text{recurrence of order 2}} - \underbrace{P_{n+1}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 4}}\Bigr) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\\text{of order 1}}} \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\\text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \underbrace{\frac{1}{1-x}\left(2 - \underbrace{\underbrace{P_n(x)}_{\substack{\text{recurrence}\\\text{of order 2}}} - \underbrace{P_{n+1}(x)}_{\substack{\text{recurrence}\\\text{of order 2}}}}_{\text{recurrence of order 4}}\right)}_{\text{recurrence of order 3}} = 0$$

$$\underbrace{\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \underbrace{\frac{1}{1-x}\Big(2 - \underbrace{\underbrace{P_n(x)}_{\text{recurrence of order 2}} - \underbrace{P_{n+1}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 4}}\Big)}_{\text{recurrence of order 3}}}_{\text{recurrence of order 7}} = 0$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\begin{aligned}
\mathrm{lhs}_{n+7} = {} & (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+6} \\
& + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+5} \\
& + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+4} \\
& + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+3} \\
& + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+2} \\
& + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+1} \\
& + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n}
\end{aligned}$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

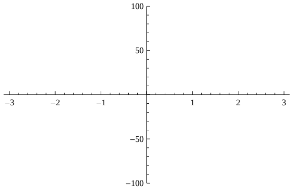
$$\begin{aligned}
\mathrm{lhs}_{n+7} = (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+6} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+5} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+4} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+3} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+2} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+1} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n}
\end{aligned}$$

Therefore the identity holds *for all* $n \in \mathbb{N}$
if and only if it holds *for* $n = 0, 1, 2, \ldots, 6$.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

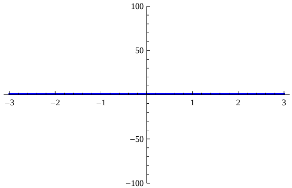
Hermite polynomials:



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

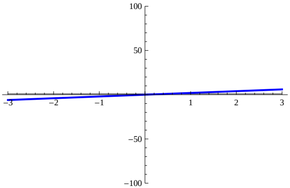
- $H_0(x) = 1$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

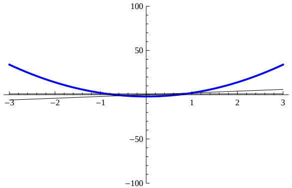
- $H_0(x) = 1$
- $H_1(x) = 2x$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

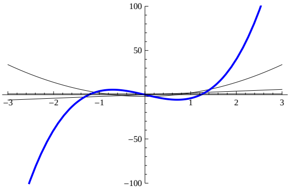
- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

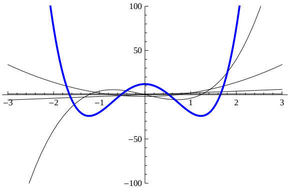
- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

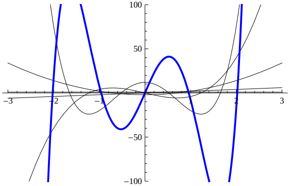
Hermite polynomials:

- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$
- $H_4(x) = 16x^4 - 48x^2 + 12$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:



- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$
- $H_4(x) = 16x^4 - 48x^2 + 12$
- $H_5(x) = 32x^5 - 160x^3 + 120x$
- $\dots$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

$$H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x)$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

$$\begin{aligned}
H_{n+2}(x) &= 2xH_{n+1}(x) - 2(n+1)H_n(x)\\
H_0(x) &= 1\\
H_1(x) &= 2x
\end{aligned}$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity between power series.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity between power series.

Consider x and y as fixed parameters.

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \, \frac{1}{n!} \, t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity between power series.

Consider x and y as fixed parameters.

Then both sides are univariate power series in t.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity between power series.

Consider x and y as fixed parameters.

Then both sides are univariate power series in t.

Idea: Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

This is an identity between power series.

Consider x and y as fixed parameters.

Then both sides are univariate power series in t.

Idea: Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\Big(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\Big)=0$$

This is an identity between power series.

Consider x and y as fixed parameters.

Then both sides are univariate power series in t.

Idea: Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

Then prove by induction that they are all zero.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

This is an identity between power series.

Consider x and y as fixed parameters.

Then both sides are univariate power series in t.

Idea: Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

Then prove by induction that they are all zero.

Then the power series is zero.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)H_n(y)}_{\text{rec. of ord. 2}} \; \frac{1}{n!} \; t^n \; - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)}_{\text{rec. of ord. 2}} \underbrace{H_n(y)}_{\text{rec. of ord. 2}} \frac{1}{n!} t^n - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\\text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\\text{ord. 2}}}}_{\text{rec. of order 4}} \frac{1}{n!}\, t^n - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}} t^n - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\substack{\text{rec. of}\\ \text{ord. 1}}}}_{\text{recurrence of order 4}}\ t^n \ - \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}} \underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}} \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}} t^n - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

differential equation of order 5

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\, t^n}_{\text{differential equation of order 5}} - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)}_{\text{rec. of ord. 2}} \underbrace{H_n(y)}_{\text{rec. of ord. 2}} \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}} t^n - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \exp \underbrace{\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}} = 0$$

- $H_n(x)H_n(y)$: rec. of order 4
- $H_n(x)H_n(y)\frac{1}{n!}$: recurrence of order 4
- $\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{1}{n!}t^n$: differential equation of order 5

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\substack{\text{rec. of}\\ \text{ord. 1}}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} \;-\; \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\substack{\text{diff.eq.}\\ \text{of ord. 1}}}\ \underbrace{\exp}_{\substack{\text{diff.eq.}\\ \text{of ord. 1}}} \underbrace{\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\;\underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\; t^n}_{\text{differential equation of order 5}} \;-\; \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\;\underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\underbrace{\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}}}_{\text{differential equation of order 1}} = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} - \underbrace{\underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\ \underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\left(\underbrace{\frac{4t(xy - t(x^2+y^2))}{1-4t^2}}_{\text{alg.eq. of order 1}}\right)}_{\text{differential equation of order 1}}}_{\text{differential equation of order 1}} = 0$$

$$\underbrace{\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} - \underbrace{\underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\ \underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\underbrace{\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 5}} = 0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)}_{\text{rec. of ord. 2}} \underbrace{H_n(y)}_{\text{rec. of ord. 2}} \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}} t^n - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \underbrace{\exp}_{\text{diff.eq. of ord. 1}} \underbrace{\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}} = 0$$

rec. of order 4 | differential equation of order 1

recurrence of order 4 | differential equation of order 1

differential equation of order 5

differential equation of order 5

$\rightsquigarrow$ recurrence equation of order 4

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

If we write $\mathrm{lhs}(t)=\sum_{n=0}^{\infty}\mathrm{lhs}_n\,t^n$, then

$$\begin{aligned}\mathrm{lhs}_{n+4} &= \tfrac{4xy}{n+4}\,\mathrm{lhs}_{n+3}+\tfrac{4(2n-2x^2-2y^2+5)}{n+4}\,\mathrm{lhs}_{n+2}\\ &\quad+\tfrac{16xy}{n+4}\,\mathrm{lhs}_{n+1}-\tfrac{16(n+1)}{n+4}\,\mathrm{lhs}_n\,.\end{aligned}$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

If we write $\mathrm{lhs}(t)=\sum_{n=0}^{\infty}\mathrm{lhs}_n\,t^n$, then

$$\begin{aligned}\mathrm{lhs}_{n+4} &= \tfrac{4xy}{n+4}\,\mathrm{lhs}_{n+3}+\tfrac{4(2n-2x^2-2y^2+5)}{n+4}\,\mathrm{lhs}_{n+2}\\ &\quad+\tfrac{16xy}{n+4}\,\mathrm{lhs}_{n+1}-\tfrac{16(n+1)}{n+4}\,\mathrm{lhs}_n\,.\end{aligned}$$

Because of $\mathrm{lhs}_0=\mathrm{lhs}_1=\mathrm{lhs}_2=\mathrm{lhs}_3=0$, we have $\mathrm{lhs}_n=0$ for all n.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

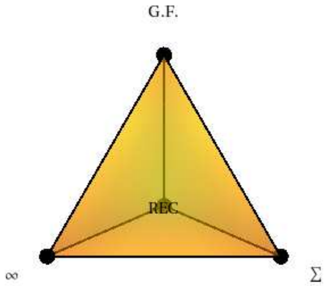
If we write $\operatorname{lhs}(t)=\sum_{n=0}^{\infty}\operatorname{lhs}_n t^n$, then

$$\begin{aligned}\operatorname{lhs}_{n+4} = {} & \tfrac{4xy}{n+4}\operatorname{lhs}_{n+3}+\tfrac{4(2n-2x^2-2y^2+5)}{n+4}\operatorname{lhs}_{n+2}\\ & +\tfrac{16xy}{n+4}\operatorname{lhs}_{n+1}-\tfrac{16(n+1)}{n+4}\operatorname{lhs}_n\,.\end{aligned}$$

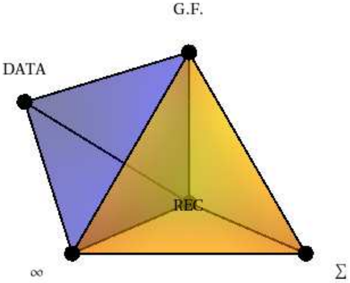
Because of $\operatorname{lhs}_0=\operatorname{lhs}_1=\operatorname{lhs}_2=\operatorname{lhs}_3=0$, we have $\operatorname{lhs}_n=0$ for all n.

This completes the proof.

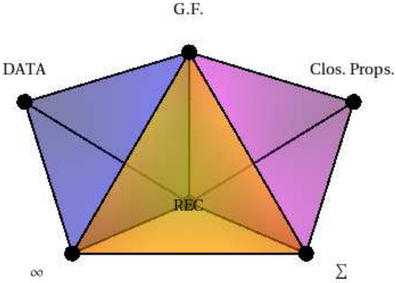
Summary.

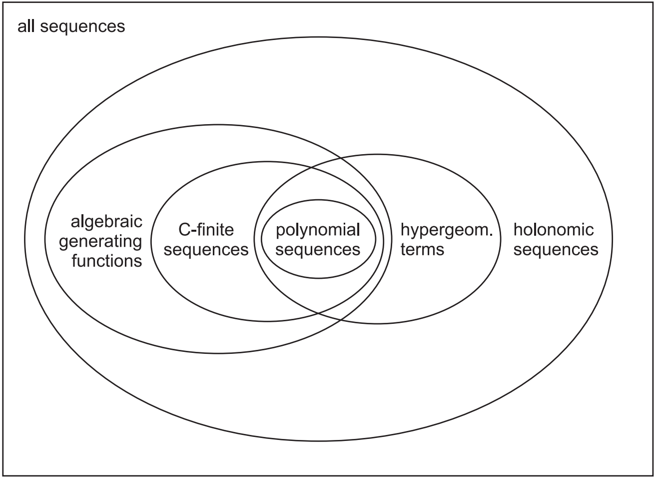


Summary.



Summary.




all sequences
algebraic
generating
functions
C-finite
sequences
polynomial
sequences
hypergeom.
terms
holonomic
sequences

The Case of Several Variables

Recall:

Recall:

- A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

Recall:

- A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

- A formal power series $a \in K[[x]]$ is called *holonomic* if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)a(x) + p_1(x)a'(x) + p_2(x)a''(x) + \cdots + p_r(x)a^{(r)}(x) = 0.$$

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are "continuous" variables ($p \in \mathbb{N}$ fixed), and

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are "continuous" variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are "continuous" variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

Examples.

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are "continuous" variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

Examples.

- $\exp(x - y)$: 2 continuous and 0 discrete variables.

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are "continuous" variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

Examples.

- $\exp(x - y)$: 2 continuous and 0 discrete variables.
- $\binom{n}{k}$: 0 continuous and 2 discrete variables.

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are "continuous" variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

Examples.

- $\exp(x - y)$: 2 continuous and 0 discrete variables.
- $\binom{n}{k}$: 0 continuous and 2 discrete variables.
- $P_n(x)$ 1 continuous and 1 discrete variable.

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are "continuous" variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

We want to *differentiate* the x_i and to *shift* the n_j:

$$\frac{\partial^5}{\partial x^5} \frac{\partial^3}{\partial y^3} f(x, y, n + 4, k + 23)$$

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

- $x_1, \ldots, x_p$ are "continuous" variables ($p \in \mathbb{N}$ fixed), and
- $n_1, \ldots, n_q$ are discrete variables ($q \in \mathbb{N}$ fixed).

We want to *differentiate* the x_i and to *shift* the n_j:

$$\frac{\partial^5}{\partial x^5}\frac{\partial^3}{\partial y^3} f(x, y, n+4, k+23)$$

Operator notation:

$$D_x^5 D_y^3 S_n^4 S_k^{23} f$$

Definition. An object $f(x_1,\dots,x_p,n_1,\dots,n_q)$ is called D-finite, if

Definition. An object $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called D-finite, if

- For every $k = 1, \ldots, p$ there exist polynomials $p_0, \ldots, p_r$ in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$, not all zero, such that

$$\left(p_0 + p_1 D_{x_k} + p_2 D_{x_k}^2 + \cdots + p_r D_{x_k}^r\right) \cdot f = 0.$$

Definition. An object $f(x_1, \dots, x_p, n_1, \dots, n_q)$ is called D-finite, if

- For every $k = 1, \dots, p$ there exist polynomials $p_0, \dots, p_r$ in the variables $x_1, \dots, x_p, n_1, \dots, n_q$, not all zero, such that

$$\bigl(p_0 + p_1 D_{x_k} + p_2 D_{x_k}^2 + \cdots + p_r D_{x_k}^r\bigr) \cdot f = 0.$$

- For every $k = 1, \dots, q$ there exist polynomials $p_0, \dots, p_r$ in the variables $x_1, \dots, x_p, n_1, \dots, n_q$, not all zero, such that

$$\bigl(p_0 + p_1 S_{n_k} + p_2 S_{n_k}^2 + \cdots + p_r S_{n_k}^r\bigr) \cdot f = 0.$$

Examples.

Examples.

- $f(x,y) = \exp(x-y)$ is D-finite because

$$\big(D_x - 1\big)\cdot f = 0 \quad \text{and} \quad \big(D_y + 1\big)\cdot f = 0.$$

Examples.

- $f(x,y) = \exp(x-y)$ is D-finite because

$$\big(D_x - 1\big)\cdot f = 0 \quad \text{and} \quad \big(D_y + 1\big)\cdot f = 0.$$

- $f(n,k) = \binom{n}{k}$ is D-finite because

$$\big((1-k+n)S_n-(n+1)\big)\cdot f = 0 \text{ and } \big((k+1)S_k+(k-n)\big)\cdot f = 0.$$

Examples.

- $f(x,y) = \exp(x-y)$ is D-finite because

$$\big(D_x - 1\big)\cdot f = 0 \quad \text{and} \quad \big(D_y + 1\big)\cdot f = 0.$$

- $f(n,k) = \binom{n}{k}$ is D-finite because

$$\big((1-k+n)S_n - (n+1)\big)\cdot f = 0 \text{ and } \big((k+1)S_k + (k-n)\big)\cdot f = 0.$$

- $f(x,n) = P_n(x)$ is D-finite because

$$\big((x^2-1)D_x^2 + 2xD_x - n(n+1)\big)\cdot f = 0 \quad \text{and}$$
$$\big((n+2)S_n^2 - (2nx-3x)S_n + (n+1)\big)\cdot f = 0$$

Counterexamples.

Counterexamples.

- $f(x,n) = \sqrt{x+n}$ is not D-finite.

Counterexamples.

- $f(x, n) = \sqrt{x + n}$ is not D-finite.

  It satisfies a differential equation in x, but no recurrence in n.

Counterexamples.

- $f(x,n)=\sqrt{x+n}$ is not D-finite.
  It satisfies a differential equation in x, but no recurrence in n.
- $f(x,n)=\binom{x}{n}$ is not D-finite.

Counterexamples.

- $f(x,n) = \sqrt{x+n}$ is not D-finite.

  It satisfies a differential equation in x, but no recurrence in n.
- $f(x,n) = \binom{x}{n}$ is not D-finite.

  It satisfies a recurrence in n, but no differential equation in x.

Counterexamples.

- $f(x,n) = \sqrt{x+n}$ is not D-finite.

  It satisfies a differential equation in x, but no recurrence in n.
- $f(x,n) = \binom{x}{n}$ is not D-finite.

  It satisfies a recurrence in n, but no differential equation in x.
- $f(n,k) = S_1(n,k)$ [Stirling numbers] is not D-finite.

Counterexamples.

- $f(x,n) = \sqrt{x+n}$ is not D-finite.

  It satisfies a differential equation in x, but no recurrence in n.

- $f(x,n) = \binom{x}{n}$ is not D-finite.

  It satisfies a recurrence in n, but no differential equation in x.

- $f(n,k) = S_1(n,k)$ [Stirling numbers] is not D-finite.

  It satisfies the recurrence

  $$\big(S_n S_k + n S_n - 1\big) \cdot f = 0,$$

  but no "pure" recurrence in S_k or S_n.

Theorem. A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.

Theorem. A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.

Example.

- Consider the equations

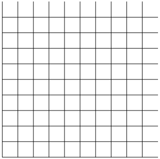
$$\big((\dots)S_n^2 + (\dots)S_n + (\dots)\big)\cdot f = 0$$
$$\big((\dots)S_k^3 + (\dots)S_k^2 + (\dots)S_k + (\dots)\big)\cdot f = 0$$

Theorem. A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.

Example.

- Consider the equations

$$\big((\dots)S_n^2 + (\dots)S_n + (\dots)\big)\cdot f = 0$$
$$\big((\dots)S_k^3 + (\dots)S_k^2 + (\dots)S_k + (\dots)\big)\cdot f = 0$$

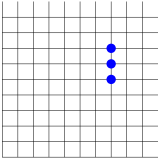


Theorem. A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.

Example.

- Consider the equations

$$\big((\dots)S_n^2 + (\dots)S_n + (\dots)\big) \cdot f = 0$$
$$\big((\dots)S_k^3 + (\dots)S_k^2 + (\dots)S_k + (\dots)\big) \cdot f = 0$$

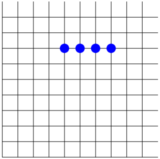


Theorem. A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.

Example.

- Consider the equations

$$\big((\dots)S_n^2 + (\dots)S_n + (\dots)\big)\cdot f = 0$$
$$\big((\dots)S_k^3 + (\dots)S_k^2 + (\dots)S_k + (\dots)\big)\cdot f = 0$$

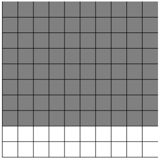


Theorem. A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.

Example.

- Consider the equations

$$\big((\ldots)S_n^2 + (\ldots)S_n + (\ldots)\big) \cdot f = 0$$
$$\big((\ldots)S_k^3 + (\ldots)S_k^2 + (\ldots)S_k + (\ldots)\big) \cdot f = 0$$

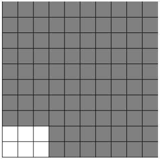


Theorem. A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.

Example.

- Consider the equations

$$\big((\dots)S_n^2 + (\dots)S_n + (\dots)\big)\cdot f = 0$$
$$\big((\dots)S_k^3 + (\dots)S_k^2 + (\dots)S_k + (\dots)\big)\cdot f = 0$$

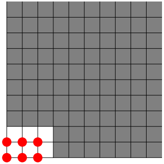


Theorem. A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.

Example.

- Consider the equations

$$\big((\dots)S_n^2 + (\dots)S_n + (\dots)\big)\cdot f = 0$$
$$\big((\dots)S_k^3 + (\dots)S_k^2 + (\dots)S_k + (\dots)\big)\cdot f = 0$$



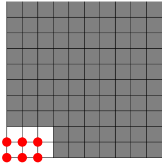
The solution is uniquely determined by

$$f(0,0), f(1,0), f(2,0), f(1,0), f(1,1), f(2,1).$$

Theorem. A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.

Example.

- Consider the equations



$$\big((\dots)S_n^2 + (\dots)S_n + (\dots)\big) \cdot f = 0$$
$$\big((\dots)S_k^3 + (\dots)S_k^2 + (\dots)S_k + (\dots)\big) \cdot f = 0$$

The solution is uniquely determined by

$$f(0,0), f(1,0), f(2,0), f(1,0), f(1,1), f(2,1).$$

Simiarly for differential equations and for systems containing mixed equations.

D-finiteness requires for every variable a pure equation.

D-finiteness requires for every variable a pure equation.

But we do not necessarily need to know them explicitly.

D-finiteness requires for every variable a pure equation.

But we do not necessarily need to know them explicitly.

It is sufficient to have a system of equations which implies the existence of a pure equation for every variable.

D-finiteness requires for every variable a pure equation.

But we do not necessarily need to know them explicitly.

It is sufficient to have a system of equations which implies the existence of a pure equation for every variable.

Example.

- $f(x,n) = P_n(x)$ satisfies

$$\big((x^2-1)D_x - (n+1)S_n + (n+1)x\big)\cdot f = 0 \quad \text{and}$$
$$\big((x^2-1)D_x^2 + 2xD_x - n(n+1)\big)\cdot f = 0.$$

D-finiteness requires for every variable a pure equation.

But we do not necessarily need to know them explicitly.

It is sufficient to have a system of equations which implies the existence of a pure equation for every variable.

Example.

- $f(x,n) = P_n(x)$ satisfies

$$\big((x^2-1)D_x - (n+1)S_n + (n+1)x\big)\cdot f = 0 \quad \text{and}$$
$$\big((x^2-1)D_x^2 + 2xD_x - n(n+1)\big)\cdot f = 0.$$

  These equations imply

$$\big((n+2)S_n^2 - (2nx-3x)S_n + (n+1)\big)\cdot f = 0.$$

Algebraic point of view:

Algebraic point of view:

Consider the operator algebra

$$A := K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q}\rangle$$

Algebraic point of view:

Consider the operator algebra

$$A := K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q}\rangle$$

Multiplication is defined here so that it is compatible with applying operators to a function.

Algebraic point of view:

Consider the operator algebra

$$A := K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q}\rangle$$

Multiplication is defined here so that it is compatible with applying operators to a function.

For L_1, L_2 and f we want $L_1 \cdot (L_2 \cdot f) = (L_1 L_2) \cdot f$.

Algebraic point of view:

Consider the operator algebra

$$A := K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q}\rangle$$

Multiplication is defined here so that it is compatible with applying operators to a function.

For L_1, L_2 and f we want $L_1 \cdot (L_2 \cdot f) = (L_1 L_2) \cdot f$.

This makes the ring slightly noncommutative.

Algebraic point of view:

Consider the operator algebra

$$A := K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q}\rangle$$

Multiplication is defined here so that it is compatible with applying operators to a function.

For L_1, L_2 and f we want $L_1 \cdot (L_2 \cdot f) = (L_1 L_2) \cdot f$.

This makes the ring slightly noncommutative. We have

$$D_{x_i} D_{x_j} = D_{x_j} D_{x_i}, \qquad D_{x_i} x_i = x_i D_{x_i} + 1,$$
$$S_{n_i} S_{n_j} = S_{n_j} S_{n_i}, \qquad S_{n_i} n_i = (n_i + 1) S_{n_i}.$$

Algebraic point of view:

Consider the operator algebra

$$A := K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q}\rangle$$

Multiplication is defined here so that it is compatible with applying operators to a function.

The set $\mathfrak{a}$ of all $L \in A$ with $L \cdot f = 0$ forms a *left ideal* in A.

Algebraic point of view:

Consider the operator algebra

$$A := K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q}\rangle$$

Multiplication is defined here so that it is compatible with applying operators to a function.

The set $\mathfrak{a}$ of all $L \in A$ with $L \cdot f = 0$ forms a *left ideal* in A.

It is called the *annihilator* of f.

Algebraic point of view:

Consider the operator algebra

$$A := K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q}\rangle$$

Multiplication is defined here so that it is compatible with applying operators to a function.

By definition, f is D-finite iff for all i, j we have

$$\mathfrak{a} \cap K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_i}\rangle \neq \{0\}$$
$$\mathfrak{a} \cap K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle S_{n_j}\rangle \neq \{0\}.$$

Algebraic point of view:

Consider the operator algebra

$$A := K(x_1, \dots, x_p, n_1, \dots, n_q)\langle D_{x_1}, \dots, D_{x_p}, S_{n_1}, \dots, S_{n_q}\rangle$$

Multiplication is defined here so that it is compatible with applying operators to a function.

By definition, f is D-finite iff for all i, j we have

$$\mathfrak{a} \cap K(x_1, \dots, x_p, n_1, \dots, n_q)\langle D_{x_i}\rangle \neq \{0\}$$
$$\mathfrak{a} \cap K(x_1, \dots, x_p, n_1, \dots, n_q)\langle S_{n_j}\rangle \neq \{0\}.$$

This is the case iff $\mathfrak{a}$ has Hilbert-dimension 0.

Closure properties. Let f and g be D-finite. Then:

Closure properties. Let f and g be D-finite. Then:

- $f + g$ is D-finite.

Closure properties. Let f and g be D-finite. Then:

- $f + g$ is D-finite.
- fg is D-finite.

Closure properties. Let f and g be D-finite. Then:

- $f + g$ is D-finite.
- fg is D-finite.
- $D_x f$ is D-finite for every continuous variable x.

Closure properties. Let f and g be D-finite. Then:

- $f + g$ is D-finite.
- fg is D-finite.
- $D_x f$ is D-finite for every continuous variable x.
- $S_n f$ is D-finite for every discrete variable n.

Closure properties. Let f and g be D-finite. Then:

- $f + g$ is D-finite.
- fg is D-finite.
- $D_x f$ is D-finite for every continuous variable x.
- $S_n f$ is D-finite for every discrete variable n.
- If $h_1, \dots, h_p$ are algebraic functions in $x_1, \dots, x_p$, free of $n_1, \dots, n_q$, then $f(h_1, \dots, h_p, n_1, \dots, n_q)$ is D-finite.

Closure properties. Let f and g be D-finite. Then:

- $f + g$ is D-finite.
- fg is D-finite.
- $D_x f$ is D-finite for every continuous variable x.
- $S_n f$ is D-finite for every discrete variable n.
- If $h_1, \ldots, h_p$ are algebraic functions in $x_1, \ldots, x_p$, free of $n_1, \ldots, n_q$, then $f(h_1, \ldots, h_p, n_1, \ldots, n_q)$ is D-finite.
- If $h_1, \ldots, h_q$ are integer-linear functions in $n_1, \ldots, n_q$, free of $x_1, \ldots, x_p$, then $f(x_1, \ldots, x_p, h_1, \ldots, h_q)$ is D-finite.

- Zero-dimensional ideals of annihilating operators for any of these can be computed from given zero-dimensional ideals of annihilating operators for f and g.

- Zero-dimensional ideals of annihilating operators for any of these can be computed from given zero-dimensional ideals of annihilating operators for f and g.
- Proofs, algorithms, and applications are the same as in the univariate case.

- Zero-dimensional ideals of annihilating operators for any of these can be computed from given zero-dimensional ideals of annihilating operators for f and g.
- Proofs, algorithms, and applications are the same as in the univariate case.
- There are also ready-to-use implementations:

- Zero-dimensional ideals of annihilating operators for any of these can be computed from given zero-dimensional ideals of annihilating operators for f and g.
- Proofs, algorithms, and applications are the same as in the univariate case.
- There are also ready-to-use implementations:
  - For Maple: `mgfun` by Chyzak, distributed together with Maple.

- Zero-dimensional ideals of annihilating operators for any of these can be computed from given zero-dimensional ideals of annihilating operators for f and g.
- Proofs, algorithms, and applications are the same as in the univariate case.
- There are also ready-to-use implementations:
  - For Maple: `mgfun` by Chyzak, distributed together with Maple.
  - For Mathematica: `HolonomicFunctions.m` by Koutschan, available from the RISC combinatorics software website.

Example.

Example.

- $f(x,n) = n!x^n \exp(x) P_{2n+3}(\sqrt{1-x^2})$

Example.

- $f(x,n) = n!x^n \exp(x) P_{2n+3}(\sqrt{1-x^2})$

In[1]:= **<< HolonomicFunctions.m**

Example.

- $f(x,n) = n!x^n \exp(x) P_{2n+3}(\sqrt{1-x^2})$

In[1]:= **<< HolonomicFunctions.m**

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) –> Type ?HolonomicFunctions for help

Example.

- $f(x,n) = n!x^n \exp(x) P_{2n+3}(\sqrt{1-x^2})$

In[1]:= **<< HolonomicFunctions.m**

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) –> Type ?HolonomicFunctions for help

In[2]:= $\mathbf{Annihilator[n!x^n Exp[x] LegendreP[2n+3, Sqrt[1-x^2]], \{Der[x], S[n]\}]}$

Example.

- $f(x,n) = n!x^n \exp(x) P_{2n+3}(\sqrt{1-x^2})$

In[1]:= $<<$ **HolonomicFunctions.m**

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) –> Type ?HolonomicFunctions for help

In[2]:= **Annihilator[$n!x^n$Exp[x]LegendreP[$2n+3$, Sqrt[$1-x^2$]], {Der[x], S[n]}]**

Out[2]= $$\Big\{(-9x^2 - \dots)D_x + (4n^2 + \dots)S_n + (13nx^4 + \dots), \\ (16n^3 + \cdots)S_n^2 + (64n^4x^3 + \dots)S_n + (16n^5x^2 + \cdots)\Big\}$$

Example.

- $f(n,k) = \binom{n}{k} + \sum_{k=0}^{n} \frac{1}{k!}$

Example.

- $f(n,k) = \binom{n}{k} + \sum_{k=0}^{n} \frac{1}{k!}$

In[3]:= $\mathbf{Annihilator[Binomial[}\boldsymbol{n, k}\mathbf{] + Sum[1/}\boldsymbol{k}\mathbf{!, \{}\boldsymbol{k}\mathbf{, 0, }\boldsymbol{n}\mathbf{\}], \{}\boldsymbol{S}\mathbf{[}\boldsymbol{n}\mathbf{], }\boldsymbol{S}\mathbf{[}\boldsymbol{k}\mathbf{]\}]}$

Example.

- $f(n,k) = \binom{n}{k} + \sum_{k=0}^{n} \frac{1}{k!}$

In[3]:= $\mathbf{Annihilator[Binomial[}\boldsymbol{n,k}\mathbf{]+Sum[1/}\boldsymbol{k}\mathbf{!,\{}\boldsymbol{k}\mathbf{,0,}\boldsymbol{n}\mathbf{\}],\{}\boldsymbol{S}\mathbf{[}\boldsymbol{n}\mathbf{],}\boldsymbol{S}\mathbf{[}\boldsymbol{k}\mathbf{]\}]}$

Out[3]= $$\Big\{(2k^2+\dots)S_k^2+(n^2+\cdots)S_k+(3kn+\cdots),\\ (n^2+\cdots)S_nS_k+(3kn+\cdots)S_n+(2kn+\cdots)S_k+(n^2+\cdots),\\ (4kn^3+\cdots)S_n^2+(n^4+\cdots)S_n+(k^2n^2+\cdots)S_k-(n^3+\cdots)\Big\}$$

What about generating functions?

What about generating functions?

If

$$f(x_1, \ldots, x_p, n_1, \ldots, n_q)$$

is D-finite in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$,

What about generating functions?

If

$$f(x_1, \ldots, x_p, n_1, \ldots, n_q)$$

is D-finite in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$, is

$$\sum_{n_1,\ldots,n_q=0}^{\infty} f(x_1, \ldots, x_p, n_1, \ldots, n_q) z_1^{n_1} z_2^{n_2} \ldots z_q^{n_q}$$

D-finite in the variables $x_1, \ldots, x_p, z_1, \ldots, z_q$?

What about generating functions?

If

$$f(x_1, \ldots, x_p, n_1, \ldots, n_q)$$

is D-finite in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$, is

$$\sum_{n_1, \ldots, n_q = 0}^{\infty} f(x_1, \ldots, x_p, n_1, \ldots, n_q) z_1^{n_1} z_2^{n_2} \ldots z_q^{n_q}$$

D-finite in the variables $x_1, \ldots, x_p, z_1, \ldots, z_q$? Not necessarily!

What about generating functions?

If

$$f(x_1, \dots, x_p, n_1, \dots, n_q)$$

is D-finite in the variables $x_1, \dots, x_p, n_1, \dots, n_q$, is

$$\sum_{n_1,\dots,n_q=0}^{\infty} f(x_1, \dots, x_p, n_1, \dots, n_q) z_1^{n_1} z_2^{n_2} \dots z_q^{n_q}$$

D-finite in the variables $x_1, \dots, x_p, z_1, \dots, z_q$? Not necessarily!

And conversely?

What about generating functions?

If

$$f(x_1,\ldots,x_p,n_1,\ldots,n_q)$$

is D-finite in the variables $x_1,\ldots,x_p,n_1,\ldots,n_q$, is

$$\sum_{n_1,\ldots,n_q=0}^{\infty} f(x_1,\ldots,x_p,n_1,\ldots,n_q) z_1^{n_1} z_2^{n_2} \ldots z_q^{n_q}$$

D-finite in the variables $x_1,\ldots,x_p,z_1,\ldots,z_q$? Not necessarily!

And conversely? Also not!

Definition: $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called *holonomic* if its generating function wrt. all discrete variables,

$$\sum_{n_1,\ldots,n_q=0}^{\infty} f(x_1, \ldots, x_p, n_1, \ldots, n_q) z_1^{n_1} z_2^{n_2} \ldots z_q^{n_q},$$

is D-finite.

Definition: $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called *holonomic* if its generating function wrt. all discrete variables,

$$\sum_{n_1,\ldots,n_q=0}^{\infty} f(x_1, \ldots, x_p, n_1, \ldots, n_q) z_1^{n_1} z_2^{n_2} \ldots z_q^{n_q},$$

is D-finite.

- If there are only continuous variables ($q = 0$), then holonomic and D-finite are the same.

Definition: $f(x_1,\dots,x_p,n_1,\dots,n_q)$ is called *holonomic* if its generating function wrt. all discrete variables,

$$\sum_{n_1,\dots,n_q=0}^{\infty} f(x_1,\dots,x_p,n_1,\dots,n_q) z_1^{n_1} z_2^{n_2} \dots z_q^{n_q},$$

is D-finite.

- If there are only continuous variables ($q=0$), then holonomic and D-finite are the same.
- If there is only one discrete variable and no continuous ones ($p=0, q=1$), then holonomic and D-finite are the same.

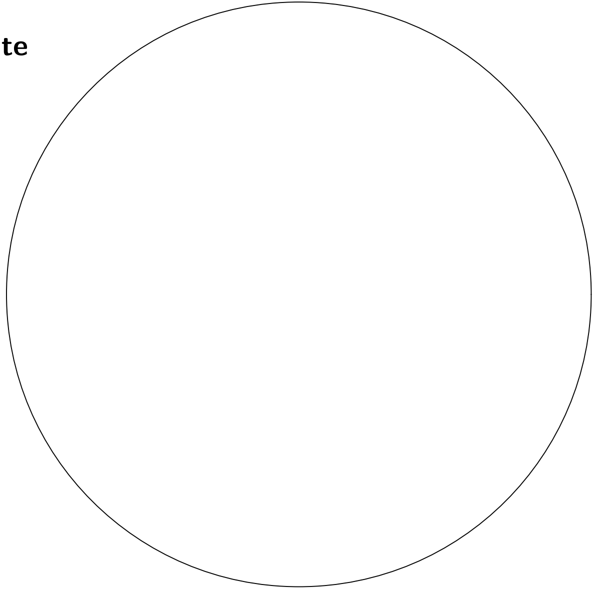
Definition: $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called *holonomic* if its generating function wrt. all discrete variables,

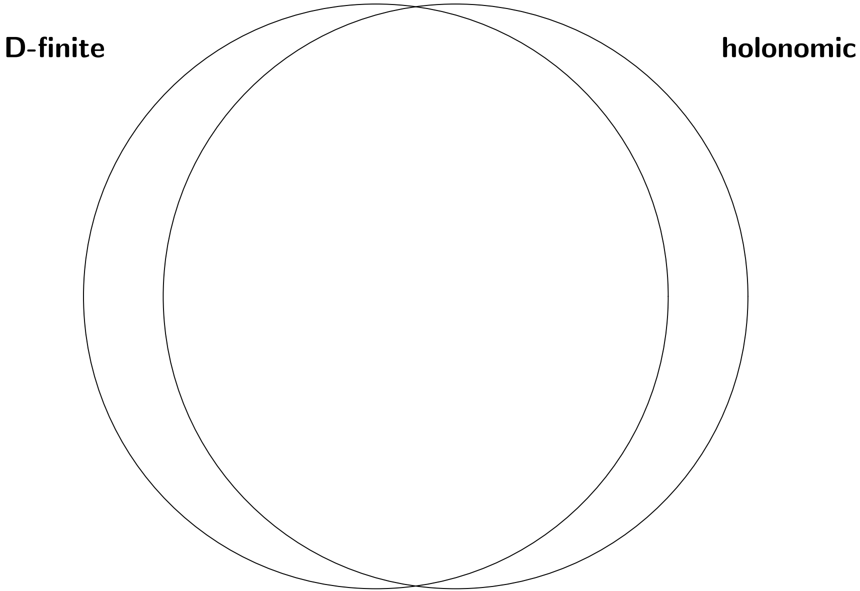
$$\sum_{n_1,\ldots,n_q=0}^{\infty} f(x_1, \ldots, x_p, n_1, \ldots, n_q) z_1^{n_1} z_2^{n_2} \ldots z_q^{n_q},$$

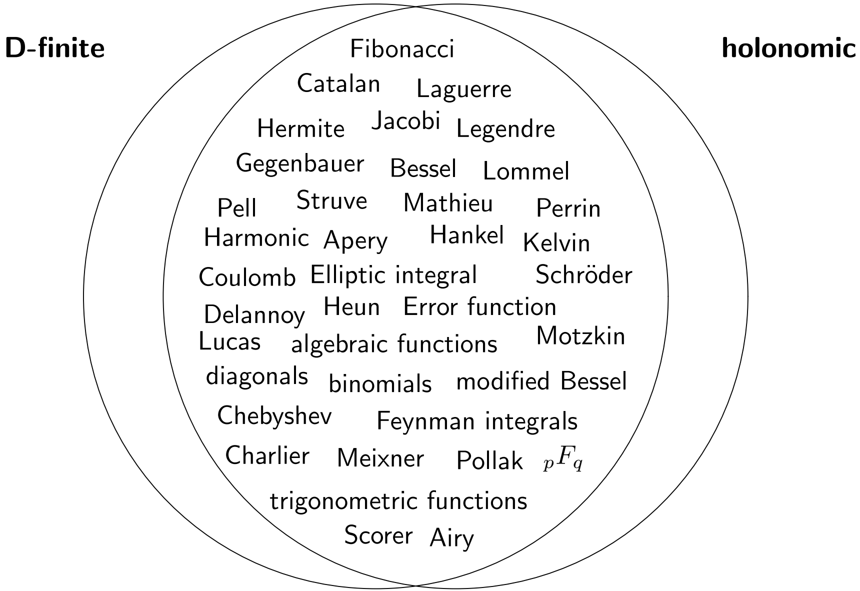
is D-finite.

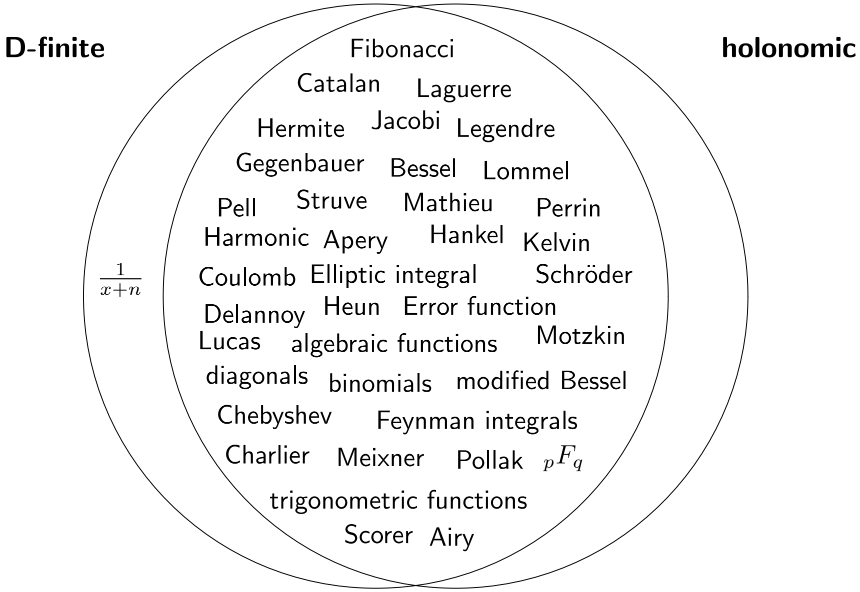
- If there are only continuous variables ($q = 0$), then holonomic and D-finite are the same.
- If there is only one discrete variable and no continuous ones ($p = 0, q = 1$), then holonomic and D-finite are the same.
- In general, holonomic and D-finite are *practically the same.*

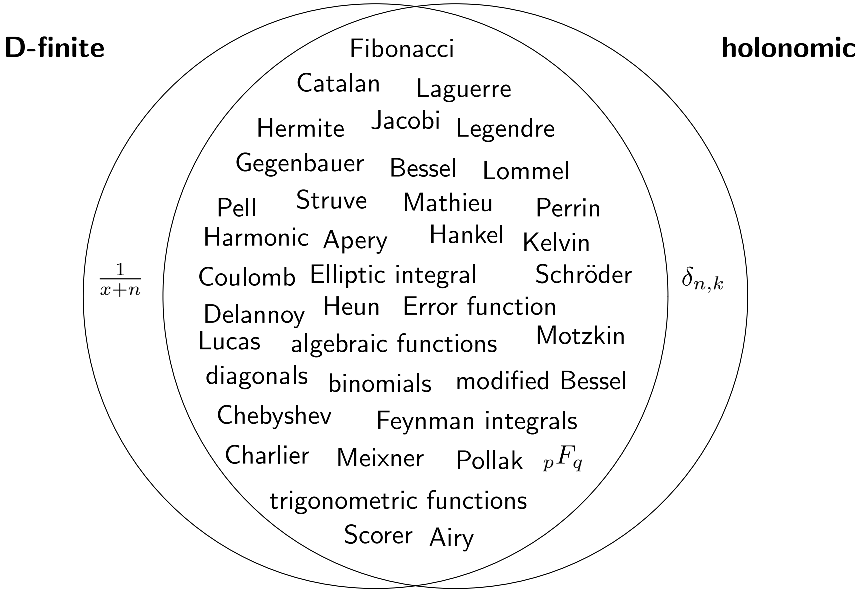
D-finite




D-finite
holonomic


D-finite
holonomic
Fibonacci
Catalan
Laguerre
Hermite
Jacobi
Legendre
Gegenbauer
Bessel
Lommel
Pell
Struve
Mathieu
Perrin
Harmonic
Apery
Hankel
Kelvin
Coulomb
Elliptic integral
Schröder
Delannoy
Heun
Error function
Lucas
algebraic functions
Motzkin
diagonals
binomials
modified Bessel
Chebyshev
Feynman integrals
Charlier
Meixner
Pollak
$_pF_q$
trigonometric functions
Scorer
Airy


D-finite
holonomic
Fibonacci
Catalan
Laguerre
Hermite
Jacobi
Legendre
Gegenbauer
Bessel
Lommel
Pell
Struve
Mathieu
Perrin
Harmonic
Apery
Hankel
Kelvin
$\frac{1}{x+n}$
Coulomb
Elliptic integral
Schröder
Delannoy
Heun
Error function
Lucas
algebraic functions
Motzkin
diagonals
binomials
modified Bessel
Chebyshev
Feynman integrals
Charlier
Meixner
Pollak
${}_pF_q$
trigonometric functions
Scorer
Airy


D-finite
holonomic
Fibonacci
Catalan
Laguerre
Hermite
Jacobi
Legendre
Gegenbauer
Bessel
Lommel
Pell
Struve
Mathieu
Perrin
Harmonic
Apery
Hankel
Kelvin
$\frac{1}{x+n}$
Coulomb
Elliptic integral
Schröder
$\delta_{n,k}$
Delannoy
Heun
Error function
Lucas
algebraic functions
Motzkin
diagonals
binomials
modified Bessel
Chebyshev
Feynman integrals
Charlier
Meixner
Pollak
${}_pF_q$
trigonometric functions
Scorer
Airy

Theorem (Summation/Integration).

Theorem (Summation/Integration).

- If f is holonomic, then so is

$$\int_{-\infty}^{\infty} f(t, x_2, \ldots, x_p, n_1, \ldots, n_q) dt,$$

provided that this integral exists.

Theorem (Summation/Integration).

- If f is holonomic, then so is

$$\int_{-\infty}^{\infty} f(t, x_2, \dots, x_p, n_1, \dots, n_q)dt,$$

  provided that this integral exists.

- If f is holonomic, then so is

$$\sum_{k=-\infty}^{\infty} f(x_1, \dots, x_p, k, n_2, \dots, n_q),$$

  provided that this sum exists.

Note the difference between indefinite and definite summation:

Note the difference between indefinite and definite summation:

Indefinite:

Definite:

Note the difference between indefinite and definite summation:

Indefinite:

$$g(n, m) = \sum_{k=0}^{n} f(k, m).$$

Definite:

Note the difference between indefinite and definite summation:

Indefinite:

$$g(n,m) = \sum_{k=0}^{n} f(k,m).$$

Definite:

$$g(m) = \sum_{k=-\infty}^{\infty} f(k,m).$$

Note the difference between indefinite and definite summation:

Indefinite:

$$g(n, m) = \sum_{k=0}^{n} f(k, m).$$

Sum and summand have the same number of variables.

Definite:

$$g(m) = \sum_{k=-\infty}^{\infty} f(k, m).$$

Note the difference between indefinite and definite summation:

Indefinite:

$$g(n,m)=\sum_{k=0}^{n} f(k,m).$$

Sum and summand have the same number of variables.

Definite:

$$g(m)=\sum_{k=-\infty}^{\infty} f(k,m).$$

The sum has one variable less than the summand.

Note the difference between indefinite and definite summation:

Indefinite:

$$g(n,m) = \sum_{k=0}^{n} f(k,m).$$

Sum and summand have the same number of variables.

$$\Downarrow$$

easy

Definite:

$$g(m) = \sum_{k=-\infty}^{\infty} f(k,m).$$

The sum has one variable less than the summand.

Note the difference between indefinite and definite summation:

Indefinite:

$$g(n,m) = \sum_{k=0}^{n} f(k,m).$$

Sum and summand have the same number of variables.

$\Downarrow$

easy

Definite:

$$g(m) = \sum_{k=-\infty}^{\infty} f(k,m).$$

The sum has one variable less than the summand.

$\Downarrow$

hard

Examples.

Examples.

- $f(n) = \sum_{k=0}^{n} 4^k \binom{n}{k}^2$ satisfies

$$\bigl((n+2)S_n^2 - (10n+15)S_n + (9n+9)\bigr)f = 0.$$

Examples.

- $f(n) = \sum_{k=0}^{n} 4^k \binom{n}{k}^2$ satisfies

$$\big((n+2)S_n^2 - (10n+15)S_n + (9n+9)\big) f = 0.$$

- $f(x) = \int_0^\infty t^2 \sqrt{t+1} \exp(-xt^2) dt$ satisfies

$$\big(16x^2 D_x^3 + (16x^2 + 96x) D_x^2 + (72x + 99) D_x + 48\big) f = 0.$$

Examples.

- $f(n) = \sum_{k=0}^{n} 4^k \binom{n}{k}^2$ satisfies
$$\big((n+2)S_n^2 - (10n+15)S_n + (9n+9)\big)f = 0.$$
- $f(x) = \int_0^\infty t^2\sqrt{t+1}\exp(-xt^2)dt$ satisfies
$$\big(16x^2D_x^3 + (16x^2+96x)D_x^2 + (72x+99)D_x + 48\big)f = 0.$$
- $f(x,t) = \sum_{n=0}^{\infty} P_n(t)x^n$ satisfies
$$\big((x^2-2tx+1)D_t - x\big)f = 0 \text{ and}$$
$$\big((x^2-2tx+1)D_x + (x-t)\big)f = 0.$$

Examples.

- $f(n) = \sum_{k=0}^{n} 4^k \binom{n}{k}^2$ satisfies

$$\big((n+2)S_n^2 - (10n+15)S_n + (9n+9)\big) f = 0.$$

- $f(x) = \int_0^\infty t^2\sqrt{t+1}\exp(-xt^2)dt$ satisfies

this is the generating function of $P_n(t)$

$$\ldots 96x)D_x^2 + (72x+99)D_x + 48\big) f = 0.$$

- $f(x,t) = \sum_{n=0}^{\infty} P_n(t)x^n$ satisfies

$$\big((x^2 - 2tx + 1)D_t - x\big) f = 0 \text{ and}$$
$$\big((x^2 - 2tx + 1)D_x + (x-t)\big) f = 0.$$

Examples.

- $f(n)=\int_0^1\int_0^1 \frac{w^{-1-\epsilon/2}(1-z)^{\epsilon/2}z^{-\epsilon/2}}{(z+w-wz)^{1-\epsilon}}(1-w^{n+1}-(1-w)^{n+1})dw\,dz$ satisfies

$$\big((8\epsilon n^7+\cdots)S_n^3-(24\epsilon n^7+\cdots)S_n^2 \\ -(24\epsilon n^7+\cdots)S_n+(8\epsilon n^7+\cdots)\big)f=0.$$

Examples.

- $f(n) = \int_0^1 \int_0^1 \frac{w^{-1-\epsilon/2}(1-z)^{\epsilon/2} z^{-\epsilon/2}}{(z+w-wz)^{1-\epsilon}} (1-w^{n+1}-(1-w)^{n+1}) dw\, dz$ satisfies

$$\big((8\epsilon n^7 + \cdots)S_n^3 - (24\epsilon n^7 + \cdots)S_n^2 \\ - (24\epsilon n^7 + \cdots)S_n + (8\epsilon n^7 + \cdots)\big) f = 0.$$

- $f(t,n) = \oint \frac{1}{\sqrt{1-2zt+z^2}} z^{-n-1} dz$ satisfies

$$\big((t^2-1)D_t - (n+1)S_n + t(n+1)\big) f = 0 \text{ and}$$
$$\big((1-t^2)D_t^2 - 2tD_t + n(n+1)\big) f = 0.$$

Examples.

- $f(n) = \int_0^1 \int_0^1 \frac{w^{-1-\epsilon/2}(1-z)^{\epsilon/2} z^{-\epsilon/2}}{(z+w-wz)^{1-\epsilon}} (1-w^{n+1}-(1-w)^{n+1}) dw\, dz$ satisfies

  this is the coefficient of z^n in the series expansion of $1/\sqrt{1-2zt+z^2}$ ("ungfun")

  $\ldots n^7 + \cdots)S_n^2$

  $\ldots + (8\epsilon n^7 + \cdots)\big)f = 0.$

- $f(t, n) = \oint \frac{1}{\sqrt{1-2zt+z^2}} z^{-n-1} dz$ satisfies

$$\big((t^2-1)D_t - (n+1)S_n + t(n+1)\big)f = 0 \text{ and}$$
$$\big((1-t^2)D_t^2 - 2tD_t + n(n+1)\big)f = 0.$$

How does this work?

How does this work?

Basic principle: Assume we have $f(x,0) = f(x,1) = 0$ and we want to find an equation for $F(x) = \int_0^1 f(x,y)dy$.

How does this work?

Basic principle: Assume we have $f(x,0) = f(x,1) = 0$ and we want to find an equation for $F(x) = \int_0^1 f(x,y)dy$.

Suppose f satisfies an equation of the form

$$a(x)f + b(x)D_x f + c(x)D_x^2 f = D_y\big(h(x,y)f\big)$$

How does this work?

Basic principle: Assume we have $f(x,0) = f(x,1) = 0$ and we want to find an equation for $F(x) = \int_0^1 f(x,y)dy$.

Suppose f satisfies an equation of the form

$$a(x)f + b(x)D_x f + c(x)D_x^2 f = D_y\big(h(x,y)f\big)$$

Then integrating both sides gives

How does this work?

Basic principle: Assume we have $f(x,0) = f(x,1) = 0$ and we want to find an equation for $F(x) = \int_0^1 f(x,y)dy$.

Suppose f satisfies an equation of the form

$$a(x)f + b(x)D_x f + c(x)D_x^2 f = D_y\big(h(x,y)f\big)$$

Then integrating both sides gives

$$a(x)F(x) + b(x)D_x F(x) + c(x)D_x^2 F(x) = 0$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$.

Example.

- $f(t,x) = t^2\sqrt{t+1}\,\exp(-xt^2)$.

$$\big(2t(t+1)D_t + (4t^3x + 4t^2x - 5t - 4)\big)f = 0,$$
$$\big(D_x + t^2\big)f = 0.$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$.

$$\big(2t(t+1)D_t + (4t^3x + 4t^2x - 5t - 4)\big)f = 0,$$
$$\big(D_x + t^2\big)f = 0.$$

$\Longrightarrow$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$.

$$\big(2t(t+1)D_t + (4t^3x + 4t^2x - 5t - 4)\big)f = 0,$$
$$\big(D_x + t^2\big)f = 0.$$

$$\implies \big(16x^2D_x^3 + (16x^2+96x)D_x^2 + (72x+99)D_x + 48\big)f$$
$$= D_t\big(-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)f\big)$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$.

$$\big(2t(t+1)D_t + (4t^3x + 4t^2x - 5t - 4)\big)f = 0,$$
$$\big(D_x + t^2\big)f = 0.$$

$$\Longrightarrow \overbrace{\big(16x^2D_x^3 + (16x^2 + 96x)D_x^2 + (72x + 99)D_x + 48\big)f}^{\text{“Telescoper”: free of } t}$$
$$= D_t\big(-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)f\big)$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\,\exp(-xt^2)$.

$$\big(2t(t+1)D_t + (4t^3x + 4t^2x - 5t - 4)\big)f = 0,$$
$$\big(D_x + t^2\big)f = 0.$$

$$\Longrightarrow \overbrace{\big(16x^2D_x^3 + (16x^2+96x)D_x^2 + (72x+99)D_x + 48\big)f}^{\text{"Telescoper": free of } t}$$
$$= D_t\big(\underbrace{-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)}_{\text{"Certificate"}}f\big)$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$. $F(x) = \int_0^\infty f(x,t)dt$

$$\big(2t(t+1)D_t + (4t^3x + 4t^2x - 5t - 4)\big)f = 0,$$
$$\big(D_x + t^2\big)f = 0.$$

$$\Longrightarrow \overbrace{\big(16x^2D_x^3 + (16x^2 + 96x)D_x^2 + (72x + 99)D_x + 48\big)f}^{\text{“Telescoper”: free of } t}$$
$$= D_t\big(\underbrace{-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)}_{\text{“Certificate”}}f\big)$$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$. $F(x) = \int_0^\infty f(x,t)dt$

$$\big(2t(t+1)D_t + (4t^3x + 4t^2x - 5t - 4)\big) f = 0,$$
$$\big(D_x + t^2\big) f = 0.$$

$$\Longrightarrow \overbrace{\big(16x^2D_x^3 + (16x^2+96x)D_x^2 + (72x+99)D_x + 48\big)f}^{\text{“Telescoper”: free of } t}$$
$$= D_t\big(\underbrace{-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)}_{\text{“Certificate”}}f\big)$$

$\Longrightarrow$

Example.

- $f(t,x) = t^2\sqrt{t+1}\exp(-xt^2)$. $F(x) = \int_0^\infty f(x,t)dt$

$$\big(2t(t+1)D_t + (4t^3x + 4t^2x - 5t - 4)\big)f = 0,$$
$$\big(D_x + t^2\big)f = 0.$$

$$\Longrightarrow \overbrace{\big(16x^2D_x^3 + (16x^2 + 96x)D_x^2 + (72x + 99)D_x + 48\big)f}^{\text{“Telescoper”: free of } t}$$
$$= D_t\big(\underbrace{-2(4t^5x - 4t^3x - 9t^3 - t^2 + 8t)}_{\text{“Certificate”}}f\big)$$

$$\Longrightarrow \big(16x^2D_x^3 + (16x^2 + 96x)D_x^2 + (72x + 99)D_x + 48\big)F = 0$$

How to construct a creative telescoping relation?

How to construct a creative telescoping relation?

There are algorithms for this task.

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology
- Algorithms based on linear algebra

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology
- Algorithms based on linear algebra
- Chyzak's algorithm (generalizing Zeilberger's algorithm)

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology
- Algorithms based on linear algebra
- Chyzak's algorithm (generalizing Zeilberger's algorithm)
- Takayama's algorithm

How to construct a creative telescoping relation?

There are algorithms for this task.

- Algorithms based on Gröbner basis technology
- Algorithms based on linear algebra
- Chyzak's algorithm (generalizing Zeilberger's algorithm)
- Takayama's algorithm

Depending on the problem at hand, any of these algorithms may be much more efficient than the others.

Summary and Outlook

- We want to solve problems in discrete mathematics using computer algebra.

- We want to solve problems in discrete mathematics using computer algebra.
- More precisely: We want to prove, discover, or simplify statements about infinite sequences.

- We want to solve problems in discrete mathematics using computer algebra.
- More precisely: We want to prove, discover, or simplify statements about infinite sequences.
- The concrete tetrahedron:
  - Symbolic sums
  - Recurrence equations
  - Generating functions
  - Asymptotic estimates

- We want to solve problems in discrete mathematics using computer algebra.
- More precisely: We want to prove, discover, or simplify statements about infinite sequences.
- The concrete tetrahedron:
  - Symbolic sums
  - Recurrence equations
  - Generating functions
  - Asymptotic estimates
- Classes of infinite sequences:
  - Polynomial sequences
  - C-finite sequences
  - Hypergeometric terms
  - Algebraic generating functions
  - Holonomic sequences

Topics of ongoing research:

Topics of ongoing research:

- Find more efficient algorithms

Topics of ongoing research:

- Find more efficient algorithms
- Find algorithms for larger classes

Topics of ongoing research:

- Find more efficient algorithms
- Find algorithms for larger classes
- Produce practical implementations

Topics of ongoing research:

- Find more efficient algorithms
- Find algorithms for larger classes
- Produce practical implementations

- Apply the techniques to problems to other people's problems, e.g., in combinatorics, partition theory, numerical analysis (Pillwein), particle physics (Schneider), . . .

Topics of ongoing research:

- Find more efficient algorithms
- Find algorithms for larger classes
- Produce practical implementations

- Apply the techniques to problems to other people's problems, e.g., in combinatorics, partition theory, numerical analysis (Pillwein), particle physics (Schneider), ...

Ideally, any piece of research on one of these sides will also stimulate interesting developments on the other.

Topics of ongoing research:

Most of the present research focusses on the multivariate case.

Topics of ongoing research:

Most of the present research focusses on the multivariate case.

Algorithms for the univariate case can already be considered folklore.

Topics of ongoing research:

Most of the present research focusses on the multivariate case.

Algorithms for the univariate case can already be considered folklore.

Rule of thumb:

Topics of ongoing research:

Most of the present research focusses on the multivariate case.

Algorithms for the univariate case can already be considered folklore.

Rule of thumb:

- If you can solve a problem with computer algebra for univariate sequences, I will probably claim that there is no reason to solve it by other means.

Topics of ongoing research:

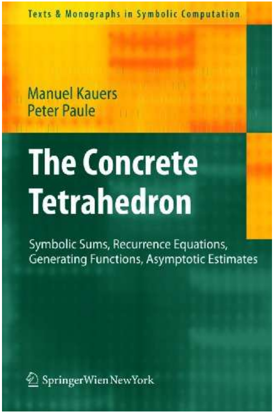
Most of the present research focusses on the multivariate case.

Algorithms for the univariate case can already be considered folklore.

Rule of thumb:

- If you can solve a problem with computer algebra for univariate sequences, I will probably claim that there is no reason to solve it by other means.
- If you can solve a problem only with computer algebra for multivariate sequences, I will probably urge you to write an article about it.

Further reading:



Further listening:

- Peter Paule's slot on January 25 in this lecture series
- The course "Analytic Combinatorics" taught by Veronika Pillwein