FUNDAMENTAL
of Numerical Analysis and Symbolic Computation
WS 2012 / 2013

Two Lectures on Discontinuous Galerkin (DG) Methods

1. Derivation of the DG Variational Formulation

- Let us consider the model problem

(1) $$\boxed{\begin{array}{l} \text{Given } f \in L_2(\Omega), \text{ find } u \in V = H^1(\Omega) \text{ such that} \\ -\operatorname{div}(a \nabla u) = f \text{ in } \Omega \subset \mathbb{R}^d \quad (f \wedge \mathrm{Lip}, d = 2,3) \\ \qquad\qquad u = g \text{ on } \Gamma = \partial\Omega \\ \text{wlg}: d = 2,\ a = 1,\ g = 0. \end{array}}$$

- Solvability: = folklore ($\to$ NuPDE) !
  Due to Lax-Milgram, there exists a unique weak solution $u \in V_g = V_0 = \overset{\circ}{H}{}^1(\Omega) \subset V = H^1(\Omega)$:

$$(1)_{VF} \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in \overset{\circ}{H}{}^1(\Omega)$$

$$a(u,v) = (f,v)_{L_2(\Omega)} \quad \forall v \in V_0$$

$$\langle Au, v \rangle = \langle F, v \rangle \quad \forall v \in V_0$$

$$Au = F \quad \text{in } V_0^* = H^{-1}(\Omega)$$

$$u = u - \tau J(Au - F) \quad \text{in } V_0$$

$J : V_0^* \to V_0$ – Riesz-Iso

$\langle \cdot, \cdot \rangle : V_0^* \times V_0 \to \mathbb{R}$ – duality product

Proof: of $\exists !$ ... by Banach's fix point theorem. $\square$

- **Exercise 1:** Let $u \in V_0$ be the weak solution of $(1)_{VF}$. Show that $\nabla u \in H(\operatorname{div}) := \{v \in [L_2(\Omega)]^d : \operatorname{div} v \in L_2(\Omega)\}$ and $\operatorname{div}(\nabla u) = f \in L_2(\Omega)$!
  Hence, $\exists$ trace $\gamma_n \nabla u :\stackrel{\cdot}{=} \nabla u \cdot n \in H^{-1/2}(\Gamma)$!

- **A trace and an inverse trace theorem:**
  Define the trace operator $\gamma_t : D(\overline{\Omega}) \to D(\Gamma)$ by
  $$\gamma_t u := u|_\Gamma .$$
  If Ω is a $C^{k-1,1}$ domain, and if $1/2 < s \le k$, the γ has a unique closure to a $\exists$ lin. operator

  (trace th.) $\quad \gamma_t : H^s(\Omega) \rightleftarrows H^{s-1/2}(\Gamma) : \gamma_e \quad$ (inv. trace th.),

  and this (operator) closure (extension) has a continuous right inverse γ_e (= inverse trace theorem), i.e
  $$\gamma_t \gamma_e u = u \quad \forall u \in H^{s-1/2}(\Gamma).$$
  Proof: [McLean] = [2], p. 102 $\square$

- **Special cases:**
  1. $s = 1 = k$, $\underbrace{\Omega \in C^{0,1}}$: $\|\gamma_t u\|_{H^{1/2}(\Gamma)} \le c_t \|u\|_{H^1(\Omega)} \; \forall u \in H^1(\Omega)$ (trace operator)
     $\|\gamma_e u\|_{H^1(\Omega)} \le c_e \|u\|_{H^{1/2}(\Gamma)} \; \forall u \in H^{1/2}(\Gamma)$ (extension operator)
  2. $s = \frac{1}{2} + \varepsilon \le k = 1$, : $\|\gamma_t u\|_{H^\varepsilon(\Gamma)} \le c_t \|u\|_{H^{1/2+\varepsilon}(\Gamma)} \; \forall u \in H^{1/2+\varepsilon}(\Omega)$
     BUT: for $\varepsilon = 0$, the trace theorem is not true !
  3. $\nabla u \in H^{3/2+\varepsilon}(\Omega) \Rightarrow \exists\, \gamma_n \nabla u \in H^\varepsilon(\Gamma) \subset L_2(\Gamma)$:

     $(2)_\varepsilon \qquad \|\gamma_n \nabla u\|_{L_2(\Gamma)} \le \|\gamma_n \nabla u\|_{H^\varepsilon(\Gamma)} \lesssim \|u\|_{H^{3/2+\varepsilon}(\Omega)}$
  4. For $\varepsilon = 1/2$, we get

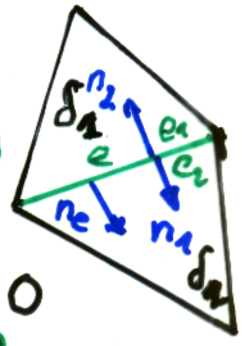
     $(2)_{1/2} \qquad \|\gamma_n \nabla u\|_{L_2(\Gamma)} := \sqrt{\int_\Gamma \left(\frac{\partial u}{\partial n}\right)^2 ds_x} \lesssim \|u\|_{H^2(\Omega)} \quad \forall u \in H^2(\Omega).$

■ "Continuity" properties in the corresponding trace spaces:

u : $(1)_F$ and $\gamma_n u := \frac{\partial u}{\partial n}$ are "continuous" across interfaces:

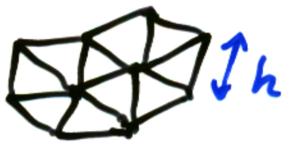
1) $[\![u]\!]_e := \underset{u|_{e_1}}{u_1} n_1 + \underset{u|_{e_2}}{u_2} n_2 = \underbrace{(u_1 - u_2)}_{[u]_e} n_e = 0$ ← $u \in H^1(\Omega)$

$[u]_e \, n_e$, $\in H^{1/2}(e)$



2) $[\nabla u]_e := \nabla u_1 \cdot n_1 + \nabla u_2 n_2 = \underbrace{(\nabla u_1 - \nabla u_2) \cdot n_e}_{\in H^{-1/2}(e)} = 0$ ← $\nabla u \in H(\mathrm{div})$

$u \in H^s(\Omega) \xrightarrow[s > 3/2]{\text{trace th.}} \in L_2(e)$



■ DG - Notations:

Let $\mathcal{T}_h = \{\delta_r : r \in R_h\}$ be a <u>regular</u> triangulation of Ω.
For $s > 0$, we define the broken Sobolev-spaces

$$H^s(\mathcal{T}_h) := \{ v \in L_2(\Omega) : v|_\delta \in H^s(\delta) \; \forall \delta \in \mathcal{T}_h \}$$

with $\|v\|^2_{H^s(\delta)} := \sum_{\delta \in \mathcal{T}_h} \|v\|^2_{H^s(\delta)} = (v, v)_{H^s(\mathcal{T}_h)}$,

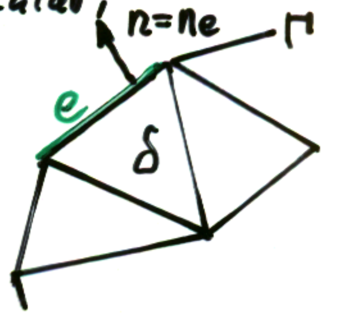
$$(u, v)_{H^s(\mathcal{T}_h)} := \sum_{\delta \in \mathcal{T}_h} (u, v)_{H^s(\delta)},$$

$$\|u\|^2_{H^s(\delta)} = \|u\|^2_{H^k(\delta)} + \sum_{|\alpha| = k} \int_\delta \int_\delta \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{d + 2\varepsilon}} \, dx\, dy$$

$s = k + \varepsilon$, $\varepsilon \in (0,1)$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = \alpha_1 + \ldots + \alpha_d$

Furthermore, we define

- the jumps (differences)
  $[\![v]\!]_e := v_1 n_1 + v_2 n_2 = \underbrace{(v_1 - v_2)}_{[v]_e} n_e$ – vector, $e \in \mathcal{E}_h$
  $[v]_e$ – scalar, inner edge
- the avarages (mean values)
  $\{\nabla v\}_e := \frac{1}{2}(\nabla v_1 + \nabla v_2)$ – vector
- $e \in \partial\mathcal{E}_h := \bar{\mathcal{E}}_h \setminus \mathcal{E}_h$ $\quad (e \in \partial\delta \cap \Gamma)$
  $[\![v]\!]_e := v n_e = v n$, $\{\nabla v\}_e := \nabla v$



■ DG - Formulation of (1):

- Let u denote the weak solution of (1), and $v \in H^s(\mathcal{T}_h)$, $s > 3/2$, an arbitrary test function. Then we have

$$(f,v)_{L_2(\Omega)} = \sum_{\delta \in \mathcal{T}_h} (f,v)_{L_2(\delta)} = \sum_{\delta} (f,v)_\delta$$

$$= \sum_{\delta} (-\operatorname{div} \nabla u, v)_\delta \qquad \in L_2(\delta) \text{ since } u \in H^s(\delta)\ \forall \delta \in \mathcal{T}_h,\ s > 3/2$$

$$= \sum_{\delta} \left[(\nabla u, \nabla v)_\delta - (\nabla u \cdot n, v)_{\partial\delta} \right]$$

$$= \sum_{\delta} (\nabla u, \nabla v)_\delta - \sum_{\delta} (\{\nabla u\}, v \cdot n)_{\partial\delta}$$

(3) $\to$ $(3)_1$: $\displaystyle\sum_{\delta} (\nabla u \cdot n, v)_{\partial\delta} = \sum_{\delta} (\{\nabla u\}, v \cdot n)_{\partial\delta}$

$$= \sum_{\delta \in \mathcal{T}_h} (\nabla u, \nabla v)_\delta - \sum_{e \in \bar{\mathcal{E}}_h} (\{\nabla u\}, [v])_e$$

(3) $\to$ $(3)_2$: $\displaystyle\sum_{\delta} (\{\nabla u\}, v \cdot n)_{\partial\delta} = \sum_{e \in \bar{\mathcal{E}}_h} (\{\nabla u\}, [v])_e$

$[u]_e = 0$

$$= \sum_{\delta} (\nabla u, \nabla v)_\delta - \sum_{e} (\{\nabla u\}, [v])_e + \beta \sum_{e} ([u], \{\nabla v\})_e$$

$$+ \underbrace{\sum_{e} \frac{\alpha_e}{h_e} ([u], [v])_e}_{\text{penalty term}} =: \underbrace{a_{DG}(u,v)}_{\text{DG bilinear form}},$$

where $\beta = -1, 0, 1$ and α_e are some positive constants.

- **Exercise 2:**
  Show the DG-identities (3), i.e.

$$\sum_{\delta \in \mathcal{T}_h} (\nabla u \cdot n, v)_{\partial\delta} = \sum_{\delta \in \mathcal{T}_h} (\{\nabla u\}, v \cdot n)_{\partial\delta} = \sum_{e \in \bar{\mathcal{E}}_h} (\{\nabla u\}, [v])_e$$

- Let us now define the <u>DG-bilinear forms</u>

(4) $a_h(u,v) = a_{DG}(u,v) = a_{DG,h,\beta}(u,v) :=$

$$:= \sum_{\delta\in T_h} (\nabla u, \nabla v)_\delta - \sum_{e\in\bar{\mathcal{E}}_h} (\{\nabla u\}, [v])_e + \beta \sum_{e\in\bar{\mathcal{E}}_h} ([u], \{\nabla u\})_e + \sum_{e\in\bar{\mathcal{E}}_h} \frac{\alpha_0}{h} ([u],[v])_e$$

$\forall u, v \in H^s(T_h)$, $s > 3/2$, where for

$\beta = -1$: $a_h(\cdot,\cdot)$ is symmetric,

$\beta = 1$ and $\beta = 0$: $a_h(\cdot,\cdot)$ is non-symmetric!

- We can now formulate the following DG-VF:

(5) $$\boxed{\text{Find } u \in H^s(T_h) : a_h(u,v) = (f,v)_{L_2(\Omega)} \; \forall v \in H^s(T_h)}$$

that makes sense for $s > 3/2$!

■ <u>Exercise 3:</u> Prove the consistency theorem!

Let $s > 3/2$. Then the following statements are valid:

(a) Assume that the weak solution u of (1), i.e. the solution of $(1)_{VF}$ ($\exists!$), belongs to $H^s(T_h)$. Then u satisfies the DG-VF (5).

(b) Conversely, if $u \in H^1(\Omega) \cap H^s(T_h)$ satisfies the DG-VF, then u is also the solution of our VF $(1)_{VF}$.

- **DG-Scheme:**

Let us define the DG-space

$$V_K(\mathcal{T}_h) := \{ v \in L_2(\Omega) : v|_\delta \in P_K(\delta) \ \forall \delta \in \mathcal{T}_h \} \subset H^s(\mathcal{T}_h) \not\subset H^1(\Omega)$$

Then the DG-Scheme reads as follows:

(6) $\boxed{\begin{array}{l} \text{Find } u_{DG} \equiv u_h \in V_K(\mathcal{T}_h) \text{ such that} \\ a_h(u_h, v_h) = (f, v_h)_{L_2(\Omega)} \ \forall v_h \in V_K(\mathcal{T}_h) \end{array}}$

- **Remark 1:**

1. $\beta = -1$: SIPG = Symmetric Interior Penalty Galerkin
   $\beta = +1$: NIPG = Nonsymmetric IPG
   $\beta = 0$: IIPG = Incomplete IPG
2. $\exists!\, u_h$: (6) ? L&M ?
   Answer = YES: provide $V_K(\mathcal{T}_h)$ with the DG-norm $\|\cdot\|_h \equiv \|\cdot\|_{DG}$ and show
   - $V_K(\mathcal{T}_h)$-ellipticity and
   - $V_K(\mathcal{T}_h)$-boundedness

   of the DG bilinear form $a_h(\cdot,\cdot)$!

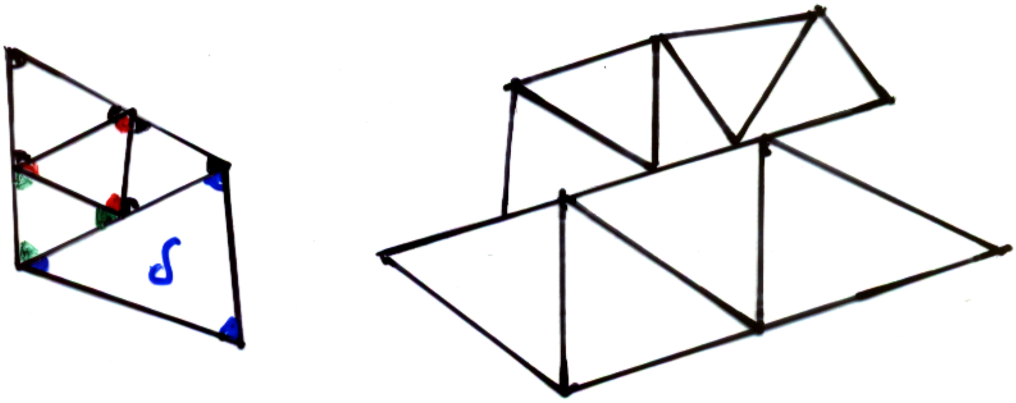
$\Longrightarrow$ see Section 2

- **Exercise 4:**

Show that the Dirichlet boundary condition $u = 0$ on Γ is incorporated in (5) resp. (6)! Derive the DG-VF resp. DG-Scheme (6) for the case of inhomogeneous Dirichlet b.c. $u = g$ on Γ and $a|_\delta = a_\delta = \text{const} > 0 \ \forall \delta \in \mathcal{T}_h$!

Remark 2: + Pros & – Cons

\+ Variational handling of hanging nodes and non-conform meshes!



\+ block ($\widehat{=}$ δ) diagonal mass matrices:
↪ well suited for time-dependent problems.

\+ natural upwinding for convection-dominated problems,

\+ conservative,

\+ better approximation properties in many practical applications, e.g. interface problems!

\+ ...

– increasing number of global dofs!
How to overcome this drawback?
⟶ DD-DG = Nitsche DG,
i.e. DG only along interfaces!
⟶ Hybridization!

– Larger stencils and non-locality due to coupling blocks ($\widehat{=} \int_{e} \dots ds_x$)

– higher regularity requirement, i.e. $u \in V_g \cap H^s(T_h)$ with some $s > 3/2$!
BUT [Cai, Ye, Zhang: SINUM, v. 49 (2011), 1761-1787]

– ...

2. $V_K(\mathcal{T}_h)$ - Ellipticity and Boundedness wrt the DG-Norm for SIPG ($\beta = -1$)

- DG-norm:

$$(7)\quad \|v\|_h^2 \equiv \|v\|_{DG}^2 := \sum_{\delta\in\mathcal{T}_h} \|\nabla v\|_{L_2(\delta)}^2 + \sum_{e\in\overline{\mathcal{E}}_h} \frac{\alpha_e}{h_e} \|[v]_e\|_{L_2(e)}^2$$

- $V_K(\mathcal{T}_h)$ - Ellipticity: For sufficiently large α_e, there exists a positive constant $\mu_1^{(DG)} = const > 0$:

$$(8)\quad a_h(v,v) \geq \mu_1^{(DG)} \|v\|_h^2 \quad \forall v \in V_K(\mathcal{T}_h).$$

- $V_K(\mathcal{T}_h)$ - Boundedness: $\exists\, \mu_2^{(DG)} = const > 0$:

$$(9)\quad |a_h(u,v)| \leq \mu_2^{(DG)} \|u\|_h \|v\|_h \quad \forall u,v \in V_K(\mathcal{T}_h)$$

- Tools for proving (8) and (9):
  1. Discrete trace Lemma: $\exists\, c = const > 0$:

$$(10)\quad \|v\|_{L_2(e)} \leq c\, h_\delta^{-1/2} \|v\|_{L_2(\delta)} \quad \forall v \in P_K(\delta)$$
$$\forall e \in \partial\delta \quad \forall \delta \in \mathcal{T}_h$$

  2. Cauchy and Youngs inequalities!

- Consequence from the Lax-Milgram-Lemma:

$\exists!\, u_h \in V_K(\mathcal{T}_h)$: (6)

- <u>Exercise 5</u>: Show that (7) defines a norm on $V_K(\mathcal{T}_h)$!

3. Convergence in the DG-Norm $\|\cdot\|_{DG} = \|\cdot\|_h$

- **Theorem 1:**

  Assume that the exact solution u to VF $(1)_{VF}$ belongs to $V_g \cap H^s(\mathcal{T}_h)$ for some $s > 3/2$.
  Then $\exists\, c = \mathrm{const} > 0$, $c \neq c(u, h)$:

$$(11) \quad \| \underset{(1)_{VF}}{u} - \underset{(6)}{u_h} \|_h \leq c\, h^{\min\{k+1, s\}-1} \|u\|_{H^s(\mathcal{T}_h)}$$

- **Main tools for the proof:**

  1. $\tilde{u}_h = \mathrm{Int}_{\tilde{V}_k(\mathcal{T}_h)}(u)$: $\|u - u_h\|_h \leq \underset{\text{OK}}{\|u - \tilde{u}_h\|_h} + \underset{?}{\|\tilde{u}_h - u_h\|_h}$
  2. Galerkin orthogonality: $a_h(u - u_h, v_h) = 0 \;\; \forall v_h \in \tilde{V}_k(\mathcal{T}_h)$
  3. trace inequality:
     $$\|v\|_{L_2(e)} \leq c\, h_\delta^{-1/2} \left(\|v\|_{L_2(\delta)} + h_\delta^{1/2+\varepsilon} |v|_{H^{1/2+\varepsilon}(\delta)} \right)$$
     $\forall v \in H^{1/2+\varepsilon}(\delta) \quad \forall e \in \partial\delta \quad \forall \delta \in \mathcal{T}_h$

4. Convergence in the L_2-Norm

- **Theorem 2:**

$$(12) \quad \|u - u_h\|_{L_2(\Omega)} \leq c\, h^{\min\{k+1, s\}} \|u\|_{H^s(\mathcal{T}_h)}$$

- **Main tool for the proof:** Nitsche-trick

- **Main References:**

[1] B. Rivière: Discontinous Galerkin Methods for Solving Elliptic and Parabolic Equations. SIAM, Philadelphia, 2008.

[2] W. McLean: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, 2000.