Algebraic Functions

K ... a field

$K[x]$ ... the ring of univariate polynomials in x with coefficients in K.

<u>Ex:</u> $5x^2 - 3x + 2 \in \mathbb{Q}[x]$

$K(x)$ ... the field of rational functions (= quotients of two polynomials) in x with coefficients in K

<u>Ex:</u> $\dfrac{5x-3}{3x+5} \in \mathbb{Q}(x)$

<u>Note:</u> rational functions are no functions!

$K(x)[Y]$ ... ring of univariate polynomials in Y with coefficients that are rational functions in x with coefficients in K.

<u>Ex:</u> $\left(\dfrac{5x+3}{7x+8}\right) \cdot Y^2 + \left(\dfrac{8x^2+x}{2x+9}\right) Y + \dfrac{x^2-1}{13x+7} \in \mathbb{Q}(x)[Y]$

Let $M \in K(x)[Y] \setminus \{0\}$ and let y be some object with $M(y) = 0$. Such an object is called an algebraic function. Let $d = \deg_Y M$ and consider the vector space

$$V := K(x) + K(x)y + \dots + K(x)y^{d-1}$$

generated by $1, y, \dots, y^{d-1}$ over $K(x)$.

(1) <u>V is a ring</u>

Indeed, for any $u, v \in V$ we can find $U, V \in K(x)[Y]$ of degree $< d$ with $u = U(y)$, $v = V(y)$. We have

$$u \cdot v = U(y) \cdot V(y) = (U \cdot V)(y).$$

By division with remainder, we can find $Q, R \in K(x)[Y]$ with

$$U \cdot V = Q \cdot M + R \quad , \quad \deg_Y R < d.$$

Then $u \cdot v = (U \cdot V)(y) = \underbrace{Q(y) \cdot \underbrace{M(y)}_{=0}}_{=0} + R(y) \in V.$

We may write $K[x)[y] := V$.

② <u>If M is square free, then V is a differential ring</u>

If $u = U(y) \in V$ with $U \in K(x)[Y]$, then the rules of differentiation force

$$D(u) = \left(\tfrac{d}{dx}U\right)(y) + \left(\tfrac{d}{dY}U\right)(y) \cdot D(y),$$

so it suffices to see that $D(y) \in V$.

Indeed, $M(y) = 0 \Rightarrow \left(\tfrac{d}{dx}M\right)(y) + \left(\tfrac{d}{dY}M\right)(y)\, D(y) = 0$

$$\Rightarrow D(y) = -\frac{\left(\tfrac{d}{dx}M\right)(y)}{\left(\tfrac{d}{dY}M\right)(y)}.$$

Since M is square free, we have
$\gcd\left(M, \tfrac{d}{dY}M\right) = 1 \Rightarrow \exists\, T, S \in K(x)[Y]:$

$$TM + S \cdot \tfrac{d}{dY}M = 1$$

$$\Rightarrow S(y) = \frac{1}{\left(\tfrac{d}{dY}M\right)(y)}$$

$$\Rightarrow D(y) = -\left(\tfrac{d}{dx}M\right)(y) \cdot S(y) \in V.$$

③ If M is irreducible, then V is a field

Indeed, if $u = U(y) \in V \setminus \{0\}$ with $U \in K(x)[Y]$ with $\deg_Y U < d$, then $\gcd(M, U) = 1$

$\Rightarrow \exists\ S, T \in K(x)[Y]: \quad SM + TU = 1$

$\Rightarrow \quad T(y) \cdot U(y) = 1 \quad \Rightarrow \quad \frac{1}{u} = T(y) \in V$

We may write $K(x)(y) := V$.

④ All elements of V are algebraic functions

Indeed, if $u \in V$, then by ①

$$1, u, u^2, \dots, u^d \in V$$

But V is by def a vector space of dimension $\leq d$, so any $d+1$ elements of V are linearly dependent

$\Rightarrow \exists\ a_0, a_1 \dots a_d \in K(x)$, not all zero,:

$$a_0 + a_1 u + \dots a_d u^d = 0$$

$\Rightarrow \exists\ A \in K(x)[Y] \setminus \{0\}: \quad A(u) = 0.$

⑤ Integral elements

$u \in V$ is called integral if $U(u) = 0$ for some $U \in K[X][Y]$ which is monic in Y.

If $u, v \in V$ are integral and $p, q \in K[X]$, then also $pu + qv$ is integral:

$$\mathcal{O}_{K[X]} := \{ u \in V \mid u \text{ is integral} \}$$

is a subring of V and a $K[X]$-module.

$\mathcal{O}_{K[X]}$ relates to V like $\mathbb{Z}$ to $\mathbb{Q}$. (Hence the name.)

An integral basis is a set of elements $\{w_1, \dots, w_m\} \subseteq V$ such that

$$\mathcal{O}_{K[X]} = K[X]w_1 \oplus K[X]w_2 \oplus \dots \oplus K[X]w_m .$$

When an integral basis is fixed, then each $u \in V$ can be written uniquely as

$$u = \frac{a_1 w_1 + \dots + a_m w_m}{b}$$

for some $a_1, \dots, a_m, b \in K[X]$ with $\gcd(a_1, \dots, a_d, b) = 1$.

⑥ If M is sqf, then y satisfies a linear ODE with polynomial coeffs

Indeed, by ② we have

$$y,\ D(y),\ D^2(y),\ \dots,\ D^d(y) \in V$$

They must be linearly dependent over $K(x)$.

Ex: $M = \ast\, y^2 + \ast\, y + \ast$

$$D(y) = \ast\, y + \ast$$
$$D^2(y) = \ast\, y + \ast$$

ansatz: $\underbrace{\begin{pmatrix} 0 & \ast & \ast \\ 1 & \ast & \ast \end{pmatrix}}_{\in K(x)^{2\times 3}} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \overset{!}{=} 0$

$\rightsquigarrow$ solve for a_0, a_1, a_2.

⑦ If $M_1, M_2 \in K(x)[Y] \setminus \{0\}$ are irreducible and y_1, y_2 are such that $M_1(y_1) = M_2(y_2) = 0$ then $y_1 + y_2$, $y_1 \cdot y_2$, $y_1 \circ y_2$ are algebraic

Consider the ideal

$$I = \langle Z - (Y_1 + Y_2),\ M_1(Y_1),\ M_2(Y_2) \rangle \subseteq K(x)[Y_1, Y_2, Z]$$

Gröbner bases theory implies that $I \cap K(x)[Z] \neq \{0\}$ (and lets us compute a nonzero element $M_3(Z)$ from M_1 and M_2).

Then
$$M_3(Z) = A \cdot (Z - (Y_1 + Y_2)) + B \cdot M_1(Y_1) + C \cdot M_2(Y_2) \quad \Big| \begin{array}{l} Z \to y_1 + y_2 \\ Y_1 \to y_1 \\ Y_2 \to y_2 \end{array}$$

$$M_3(y_1 + y_2) = A \cdot 0 + B \cdot 0 + C \cdot 0 = 0$$

The argument for $\cdot$ and $\circ$ is similar.

(8.) If $M = m_0(x) + m_1(x) Y + \dots + m_d(x) Y^d \in K[x][Y]$ with $m_0(0) = 0$, $m_1(0) \neq 0$, then y can be identified with a unique formal power series $\sum_{n=1}^{\infty} a_n x^n$

Indeed, make an ansatz $y = \sum_{n=1}^{\infty} a_n x^n$

Clearly, $M(y)$ is a FPS. It is zero iff all its coeffs are zero.

We have

$$[x^0]\, M(y) = 0 \quad \text{because} \quad m_0(0) = 0 \text{ and } a_0 = 0.$$

Next,

$$[x^1]\, M(y) = [x^1]\, m_0 + \underbrace{[x^1]\, m_1 y}_{= a_1 \cdot \underbrace{[x^0]\, m_1}_{\neq 0}} + \underbrace{[x^1]\, y^2 (\,---\,)}_{=0}$$

$$= 0 \iff a_1 = - \frac{[x^1]\, m_0}{[x^0]\, m_1}$$

In general,

$$[x^n]\, M(y) = [x^n]\, m_0 + [x^n]\, m_1 y + \;---$$

$$= a_n \cdot [x^0]\, m_1 + \underbrace{\;-----\;}_{\text{depends only on } a_0 \ldots a_{n-1}}$$

$\Rightarrow$ there is a unique choice for a_n that turns $[x^n]\, M(y)$ to zero

By induction, all terms a_n of the series are uniquely determined by M.

(8) In general, if M is irreducible and K is algebraically closed, y can be identified with d different series of the form $x^k a(x^{\frac{1}{r}})$ where $k \in \mathbb{Z}$, $r \in \mathbb{N}$, $a \in K[[x]]$

Ansatz: $y = a_\alpha x^\alpha +$ higher order terms

Idea: determine $\alpha \in \mathbb{Q}$ in such a way that $M(a_\alpha x^\alpha)$ has as lowest order coefficient a nontrivial polynomial in a_α. The roots of this polynomial are then the admissible values for a_α. Once α and a_α are known, make an ansatz $y = a_\alpha x^\alpha + a_\beta x^\beta +$ h.o.t. and proceed in the same way to find $\beta > \alpha$ and a corresponding coefficient a_β, etc.

How to ~~[illegible]~~ find good αs: Write

$$M = m_0 + m_1 Y + m_2 Y^2 + \dots + m_d Y^d$$

with
$$m_0 = \circledast x^{e_0} + \text{h.o.t.}$$
$$m_1 = \circledast x^{e_1} + \text{h.o.t.}$$
$$\vdots$$

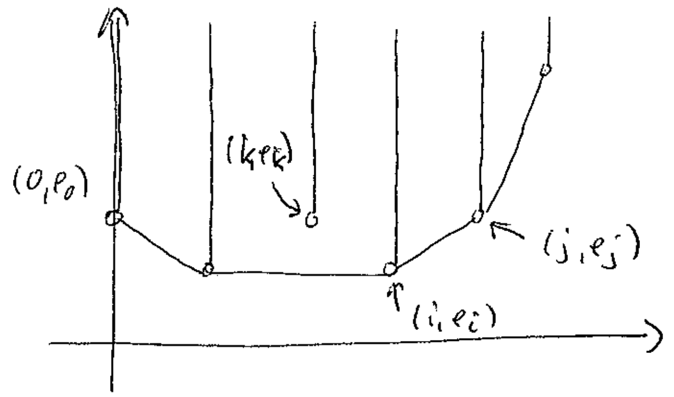
Then
$$M(a_\alpha x^\alpha) = \circledast x^{e_0} + \circledast a_\alpha x^{e_1+\alpha} + \circledast a_\alpha^2 x^{e_2+2\alpha} + \dots + \circledast a_\alpha^d x^{e_d+d\alpha} + \text{h.o.t.}$$

Need: the lowest order coefficient of $M(a_\alpha x^\alpha)$, which is a polynomial in a_α, should have at least two terms. In order to achieve this, we must choose α such that

$e_i + i\alpha = e_j + j\alpha$ for some $i \neq j$ and
$e_i + i\alpha \leq e_k + k\alpha$ for all other k.

$\alpha = -\dfrac{e_i - e_j}{i - j}$ is the negative slope of the line segment connecting (i, e_i) and (j, e_j).

$\leadsto$ The <u>Newton-Polygon</u> is defined as the convex hull of all the halflines starting at (i, e_i) for some $i = 0 \dots d$ and going upwards:

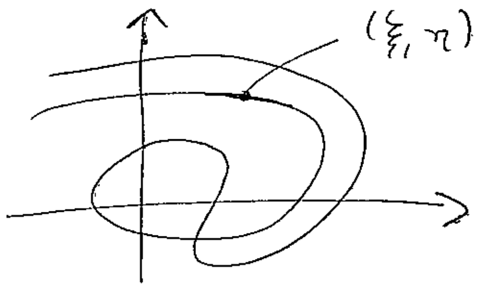


Admissible values ~~of~~ for α are the negative slopes of the nonvertical edges in the Newton polygon.

(10) Geometric Interpretation $(K = \mathbb{C})$

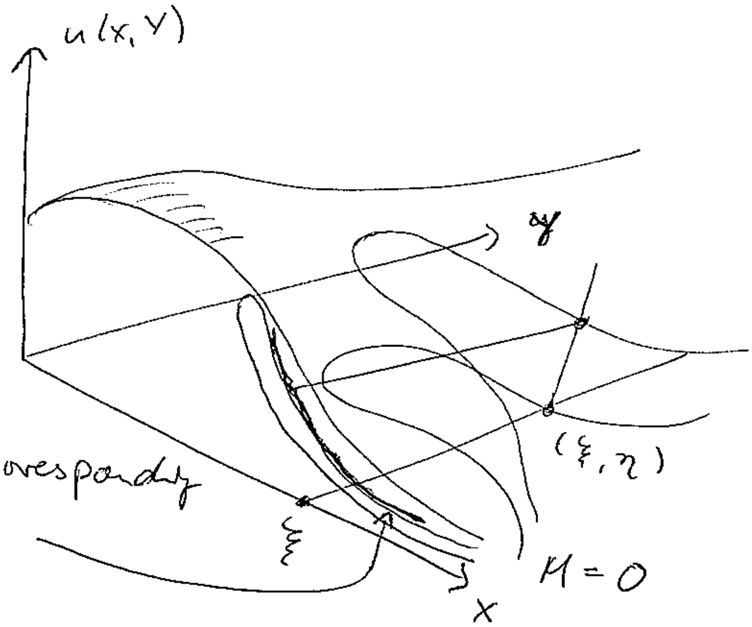
wlog $M \in K[x][Y]$ (instead of $M \in K(x)[Y]$).
For each fixed $\xi \in K$, $M|_{x=\xi} \in K[Y]$ has a finite number of roots, but possibly more than one. Function?

The set $\{(\xi,\eta) \in K^2 \mid M(\xi,\eta) = 0\}$ is a curve:



The curve defines locally a function $f: U \to \mathbb{C}$ with $f(\xi) = \eta$ for any $\xi \in K$ where $M|_{x=\xi}$ has a simple root η.
It is called a branch of the algebraic function y. The point (ξ,η) is called a place over ξ.

$u \in K(x)[y] = V$:



(11) Singular Points

ξ is called a singular point of y if $M|_{x=\xi} \in K[Y]$ has less than $\deg_Y M$ distinct roots in $\bar{K}$. This may happen for two reasons:

(1) $\deg_Y M|_{x=\xi} < \deg_Y M$ $\iff$ ξ is a root of $lc_Y M$

$\iff$ some branch of y has a pole at ξ

$\iff$ some of the series solutions of M involve terms with negative exponents. (expansion at ξ)

(2) $M|_{x=\xi}$ has a multiple root η

$\iff$ some branch of y has a branch point at (ξ, η)

$\iff$ some of the series solutions of M involve fractional exponents (expansion at ξ).

$\iff$ the vertical line through ξ is a tangent of the curve $M=0$ at (ξ,η)

$\iff$ $M(\xi,\eta) = (\frac{d}{dY}M)(\xi,\eta) = 0$

$\iff$ $\gcd(M|_{x=\xi}, (\frac{d}{dY}M)|_{x=\xi}) \neq 1$

$\iff$ ξ is a root of $res_Y(M, \frac{d}{dY}M) \in K[x]$.

(12) Asymptotics

Let $a = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[[x]]$ be an algebraic power series, say $M(x, a(x)) = 0$ for some polynomial $M \in \mathbb{C}[x, Y]$.

In general there is no nice expression for the n-th coefficient a_n. But there is always a nice expression for its asymptotic behaviour:

Let $A: U \to \mathbb{C}$ be the branch whose expansion at ~~x~~ $x=0$ is a, i.e. $A(0) = a_0$, $A'(0) = a_1, \dots$, with U being a disk centered at 0 with maximal possible radius.

Then there is some singular point ξ on the boundary of U. Suppose there is only one.

A behaves asymptotically for $x \to \xi$ like the solution $\tilde{A}$ of $\tilde{M}(x, Y) := M(\xi(1-x), Y)$ for $x \to 1$.

compute the series solutions of $\tilde{M}$ at 0. Select the one which corresponds to the branch A, so that

$$A(x) = \sum_{k=k_0}^{\infty} c_k \left(1 - \frac{x}{\xi}\right)^{k/r} \qquad (x \to \xi)$$

Let k be the smallest index such that $c_k \neq 0$ and $k/r \notin \mathbb{N}$. Then

$$a_n \sim \frac{c_k}{\Gamma(-\frac{k}{r})} \left(\frac{1}{\xi}\right)^n \cdot n^{-1-\frac{k}{r}} \qquad (n \to \infty)$$

(in the sense that $a_n \sim b_n :\Leftrightarrow \frac{a_n}{b_n} \xrightarrow{n \to \infty} 1$)

For a detailed example, see Kauers/Paule, "The Concrete Tetrahedron", Section 6.5

Let $M = 4x - 4Y^2 + 13xY^2 + 4Y^3 + 4xY^4 \in \mathbb{Q}(x)[Y]$ and let y be such that $M(y) = 0$.

Task 1 Compute $a_0(x), a_1(x), a_2(x) \in \mathbb{Q}(x)$ such that $\frac{d}{dx}y = a_0(x) + a_1(x)y + a_0(x)y^2$.

Task 2 Compute a polynomial $M' \in \mathbb{Q}(x)[Y]$ such that $M'(y^2 + xy - 1) = 0$.

Task 3 Compute a polynomial $M'' \in \mathbb{Q}(x)[Y]$ with $M''(y \circ \sqrt{1 - x^2}) = 0$.

Task 4 Compute the first few terms of the four series expansions of y.

Task 5 Compute the singular points of y.

Task 6 Determine the asymptotic behaviour of the coefficient sequence $(a_n)_{n=0}^{\infty}$ in the series expansion $\sum_{n=0}^{\infty} a_n x^n$ of the branch of y going through the place $(0, 1)$.

You are welcome to do all the calculations using computer algebra system, and to submit a transcript of your session as solution.