SCSS 2016

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Symbolic Surprises: Unexpected Findings in Combinatorics, Number Theory, and Special Functions

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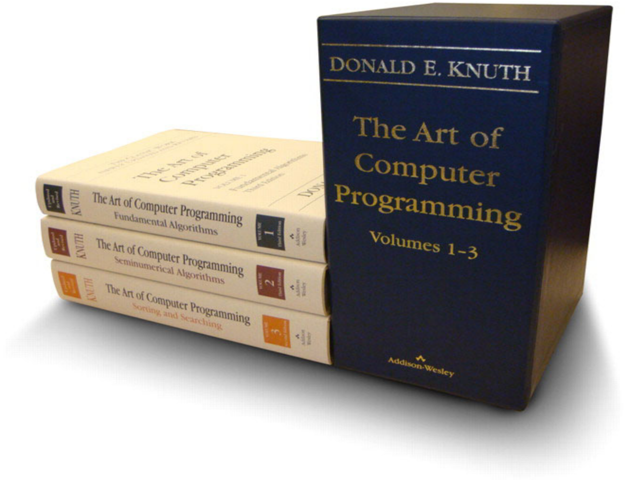
A Bit of History

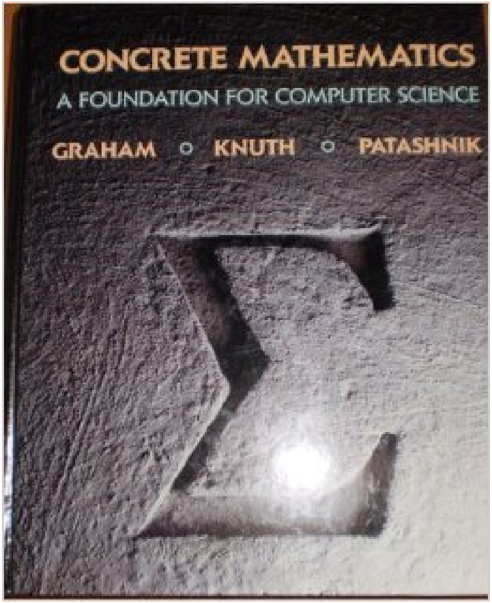
SYMBOLIC COMPUTATION in Combinatorics, Number Theory, and Special Functions

Number Theorists played a pioneering role; e.g.:
"Computers in Number Theory" (Oxford, 1969[1])

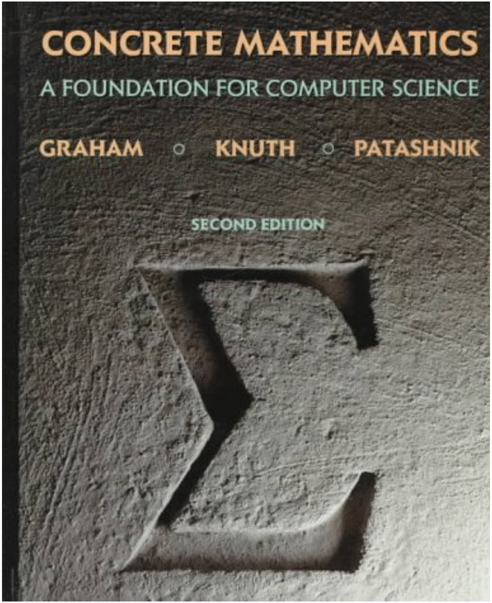
[1]taken from John B. Cosgrove's home page



Volume 1, 1st edition 1968, Exercise 1.2.6.63:
[50] Develop computer algebra programs for simplifying sums that involve binomial coefficients.



1st edition 1989; contains Gosper's algorithm (1978).



2nd edition 1994: What is the difference to the 1st edition?

Answer:

Gosper → Zeilberger:

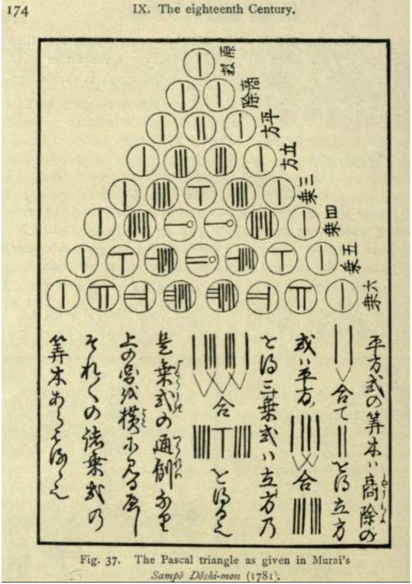


Fig. 37. The Pascal triangle as given in Murai's *Sampō Dōshi-mon* (1781).

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Web source: `https://archive.org/stream/historyofjapanes00smitiala`

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Telescoping (Gosper):

$$(-1)^k \binom{n}{k} = (-1)^k \binom{n-1}{k} - (-1)^{k-1} \binom{n-1}{k-1},$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Telescoping (Gosper):

$$(-1)^k \binom{n}{k} = (-1)^k \binom{n-1}{k} - (-1)^{k-1} \binom{n-1}{k-1} = g(k) - g(k-1),$$

$$\sum_{k=0}^{\ell} (-1)^k \binom{n}{k} = (-1)^{\ell} \binom{n-1}{\ell} - (-1)^{-1} \binom{n-1}{-1} = g(\ell) - g(-1)$$

$$= (-1)^{\ell} \binom{n-1}{\ell}.$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Telescoping (Gosper):

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$$= (-1)^\ell \binom{n-1}{\ell}.$$

We know that there are identities like

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

but $\binom{n}{k}$ does not telescope! $\rightsquigarrow$ Creative Telescoping (Zeilberger):

$$\text{Recall:} \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Creative Telescoping (Zeilberger):

$$\binom{n+1}{k} - 2\binom{n}{k} = -\binom{n}{k} + \binom{n}{k-1} = g(k) - g(k-1)$$

where

$$g(k) = -\binom{n}{k}.$$

Recall: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Creative Telescoping (Zeilberger):

$$\binom{n+1}{k} - 2\binom{n}{k} = -\binom{n}{k} + \binom{n}{k-1} = g(k) - g(k-1)$$

where

$$g(k) = -\binom{n}{k}.$$

$$\sum_{k=0}^{n+1}\left(\binom{n+1}{k} - 2\binom{n}{k}\right) = \sum_{k=0}^{n+1}\binom{n+1}{k} - 2\sum_{k=0}^{n+1}\binom{n}{k}$$

$$= g(n+1) - g(-1) = 0.$$

Recall: $$\sum_{k=0}^{n+1}\left(\binom{n+1}{k}-2\binom{n}{k}\right)=\sum_{k=0}^{n+1}\binom{n+1}{k}-2\sum_{k=0}^{n+1}\binom{n}{k}$$
$$=g(n+1)-g(-1)=0.$$

Consequently, for

$$S(n):=\sum_{k=0}^{n}\binom{n}{k}$$

one has:

$$\boxed{S(n+1)-2\,S(n)=0,\quad n\geq 0.}$$

Alternatively,

$$\text{Recall: } \sum_{k=0}^{n+1}\left(\binom{n+1}{k} - 2\binom{n}{k}\right) = \sum_{k=0}^{n+1}\binom{n+1}{k} - 2\sum_{k=0}^{n+1}\binom{n}{k}$$
$$= g(n+1) - g(-1) = 0.$$

Consequently, for

$$S(n) := \sum_{k=0}^{n}\binom{n}{k}$$

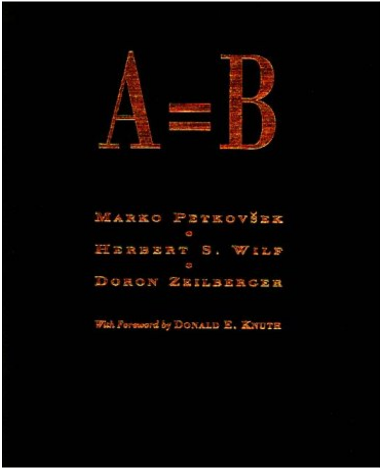
one has:

$$\boxed{S(n+1) - 2\,S(n) = 0, \quad n \geq 0.}$$

Alternatively,

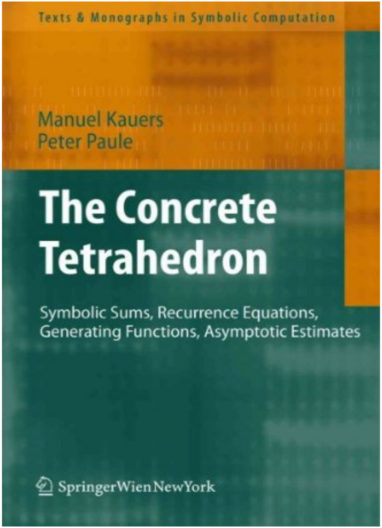
$$S(n) = 2^n, \quad n \geq 0.$$

Zeilberger's algorithm solves Knuth's [50]-problem from 1968:



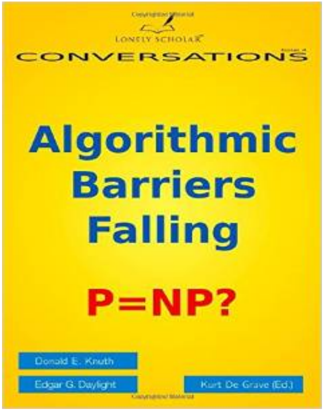
1st edition 1996

An algorithmic supplement to "Concrete Mathematics":



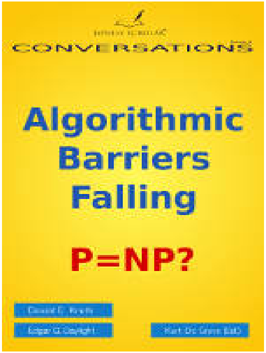
1st edition 2011

A conversation with Donald E. Knuth conducted
by Edgar G. Daylight (Paris, June 18, 2014):



Knuth: Learning how to manipulate formulas fluently, and how to see patterns in formulas instead of patterns in numbers — that's what my book "Concrete Mathematics" is essentially about.

A conversation with Donald E. Knuth conducted
by Edgar G. Daylight (Paris, June 18, 2014) [contd.]:



Edgar: Which was also the topic of Manuel Kauers this morning?

Knuth: Right. In fact, he and Peter Paule in Austria recently published a beautiful book called “The Concrete Tetrahedron”, which is sort of the sequel to ”Concrete Mathematics”.

Contents of the "Concrete Tetrahedron"

Further Concrete Surprises

Emil Artin[1]: *"...determine the number of possible products of* n *elements given in linear order. For example, the elements* a_1, a_2, a_3, a_4 *in that order yield the products* $(a_1a_2)(a_3a_4)$, $a_1(a_2(a_3a_4)))$, *etc.*

[1] Exercise 2, p.2, of "Algebra with Galois Theory", Courant LNS 15

Emil Artin[1]: *"...determine the number of possible products of n elements given in linear order. For example, the elements a_1, a_2, a_3, a_4 in that order yield the products $(a_1a_2)(a_3a_4)$, $a_1(a_2(a_3a_4)))$, etc.*

Hint. Let c_{n-1} be the number of products of $a_1, a_2, \ldots, a_n$. Find a recursion formula for c_n and use the [Lagrange] generating function

$$F(x) = c_0x + c_1x^2 + \cdots + c_{n-1}x^n + \cdots ."$$

$c_0 = 1 : (a_1)$;
$c_1 = 1 : (a_1a_2)$;
$c_2 = 2 : a_1(a_2a_3), (a_1a_2)a_3$;
$c_3 = 5 : a_1(a_2(a_3a_4)), a_1((a_2a_3)a_4), (a_1a_2)(a_3a_4),$
$(a_1(a_2a_3))a_4, ((a_1a_2)a_3)a_4$; etc.

[1] Exercise 2, p.2, of "Algebra with Galois Theory", Courant LNS 15

COMPUTER-ASSISTED GUESSING

In[1]:=

```
<< RISC`GeneratingFunctions`
```

Package GeneratingFunctions version 0.7 written by Christian Mallinger
Copyright 1996-2009, Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

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```
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Package GeneratingFunctions version 0.7 written by Christian Mallinger
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In[3]:=

```
GuessNext2Values[Li_] := Module[{rec},
  rec = GuessRE[Li, c[k], {1, 2}, {0, 3}];
  RE2L[rec[[1]], c[k], Length[Li] + 1]]
```

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In[5]:=

```
GuessNext2Values[{1, 2, 4, 8, 16}]
```

COMPUTER-ASSISTED GUESSING

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```

In[5]:=

```
GuessNext2Values[{1, 2, 4, 8, 16}]
```

Out[5]=

```
{1, 2, 4, 8, 16, 32, 64}
```

What is the idea behind?

```
GuessRE[{1, 2, 4, 8, 16}, c[k]]

{{-2 c[k] + c[1 + k] == 0, c[0] == 1}, ogf}

GuessRE[{1, 3, 6, 10, 15, 21}, c[k]]

{{(-3 - k) c[k] + (1 + k) c[1 + k] == 0, c[0] == 1}, ogf}

GuessRE[{1, 1, 2, 6, 24, 120}, c[k]]

{{(-1 - k) c[k] + c[1 + k] == 0, c[0] == 1}, ogf}

GuessRE[{1, 1, 2, 3, 5, 8}, c[k]]

{{-c[k] - c[1 + k] + c[2 + k] == 0, c[0] == 1, c[1] == 1},
```

NOTE. The first 3 sequences are hypergeometric!

NOTE. In general, there can be infinitely many recursive descriptions of given sequences!

$$\text{EXAMPLE. } c(k) := \sum_{j=1}^{k} j = 1 + 2 + \cdots + k.$$

From the recursive structure of the sum quantifier:

$$c(k) = c(k-1) + k, \ k \geq 1, \ \text{ and } \ c(0) = 1.$$

`GuessRE[...]` computed"

$$(k+1)c(k+1) - (k+3)c(k) = 0, \ k \geq 0, \ \text{ and } \ c(0) = 1.$$

Other recurrences are, for example:

$$c(k+2) - 2c(k+1) + c(k) = 1, \ k \geq 0, \ \text{ and } \ c(0) = 1, c(1) = 3,$$
$$c(k) = (k+1)(k+2)/2, \ k \geq 0; \ \text{ etc.}$$

A sequence $(c_k)_{k \geq 0}$ is called hypergeometric $:\Longleftrightarrow$

$\exists$ rational function $r(x)$ s.t. for all $k \geq 0$: $\dfrac{c_{k+1}}{c_k} = r(k)$.

Suppose we can write the rational function $r(k)$ as:

$$\frac{c_{k+1}}{c_k} = \frac{(k + a_1) \ldots (k + a_p)}{(k + b_1) \ldots (k + b_q)} \frac{z}{k + 1},$$

then:

$$c_k = c_0 \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!}.$$

NOTE. This provides a normal form representation for hypergeometric sequences!

Rising Factorials

In[6]:=

```
(a_)_k_ := Pochhammer[a, k]
```

$$(a\_)_{k\_} := \text{Pochhammer}[a, k]$$

In[7]:=

$$\{(a)_0, (a)_1, (a)_2, (a)_5\}$$

Out[7]=

$$\{1, a, a (1+a), a (1+a) (2+a) (3+a) (4+a)\}$$

Binomials

In[8]:=

$$\binom{n\_}{k\_}_* := \text{Binomial}[n, k]$$

In[9]:=

$$\binom{a}{3}_*$$

Out[9]=

$$\frac{1}{6} (-2+a) (-1+a) a$$

Back to Artin's Problem: IDENTIFY A RECURSIVE PATTERN

$c_0 = 1 : (a_1);$
$c_1 = 1 : (a_1 a_2);$
$c_2 = 2 : a_1(a_2 a_3), (a_1 a_2)a_3;$
$c_3 = 5 : a_1(a_2(a_3 a_4)), a_1((a_2 a_3)a_4), (a_1 a_2)(a_3 a_4),$
$(a_1(a_2 a_3))a_4, ((a_1 a_2)a_3)a_4$; etc.

For example,

$$c_3 = c_0\, c_2 + c_1\, c_1 + c_2\, c_0.$$

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In general, for $n \geq 1$:

$$c_n = \sum_{k=0}^{n-1} c_k\, c_{n-1-k}.$$

NOTE.

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In general, for $n \geq 1$:

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NOTE. It was relatively easy to identify this recursive pattern; however, this recurrence is not linear!

GUESSING A LINEAR RECURRENCE

In[10]:=

```
GuessRE[{1, 1, 2, 5, 14, 42}, c[k]]
```

GUESSING A LINEAR RECURRENCE

In[10]:=

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GuessRE[{1, 1, 2, 5, 14, 42}, c[k]]
```

Out[10]=

```
{{-2 (1 + 2 k) c[k] + (2 + k) c[1 + k] == 0, c[0] == 1}, ogf}
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Consequently,

$$\frac{c_{k+1}}{c_k} = \frac{\left(k + \frac{1}{2}\right) (k+1)}{(k+2)} \frac{4}{k+1};$$

i.e., for $k \geq 0$:

$$c_k = c_0 \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!} = \frac{\left(\frac{1}{2}\right)_k (1)_k}{(2)_k} \frac{4^k}{k!}$$

$$= \frac{1}{k+1} \binom{2k}{k}$$

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$$= \frac{1}{k+1} \binom{2k}{k}$$

HOW TO PROVE THIS?

RECALL: For $n \geq 1$,

$$c_n = \sum_{k=0}^{n-1} c_k \, c_{n-1-k}.$$

Using a RISC implementation of Zeilberger's "fast" algorithm we prove that our conjectured expression

$$c_n = \frac{1}{n+1} \binom{2n}{n}$$

indeed satisfies this recurrence:

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$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

indeed satisfies this recurrence:

In[11]:=

```
<< RISC`fastZeil`
```

Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright 1995-2015, Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[12]:=

$$\text{Zb}\left[\frac{1}{k+1}\binom{2k}{k}_{*}\frac{1}{n-k}\binom{2(n-1-k)}{n-1-k}_{*},\ \{k,0,n-1\},\ n,\ 1\right]$$

In[12]:=

$$\mathrm{Zb}\left[\frac{1}{k+1}\binom{2k}{k}_{*}\frac{1}{n-k}\binom{2(n-1-k)}{n-1-k}_{*},\ \{k,0,n-1\},\ n,\ 1\right]$$

```
If `-1 + n' is a natural number, then:
```

Out[12]=

$$\left\{4\,n\,\mathrm{SUM}[n]+(-2-n)\,\mathrm{SUM}[1+n]==-\frac{2\,\mathrm{Binomial}[2+2\,(-1+n),1+n]}{n},\right.$$
$$\left.-2\,(1+2\,n)\,\mathrm{SUM}[n]+(2+n)\,\mathrm{SUM}[1+n]==0\right\}$$

⇒ The homogeneous recurrence is nothing but the recurrence for the c_n: RECALL

In[12]:=

$$\text{Zb}\left[\frac{1}{k+1}\binom{2k}{k}_* \frac{1}{n-k}\binom{2(n-1-k)}{n-1-k}_*, \{k, 0, n-1\}, n, 1\right]$$

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$\Rightarrow$ The homogeneous recurrence is nothing but the recurrence for the c_n: RECALL

In[10]:=

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GuessRE[{1, 1, 2, 5, 14, 42}, c[k]]
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Out[10]=

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```
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Out[10]=

```
{{-2 (1 + 2 k) c[k] + (2 + k) c[1 + k] == 0, c[0] == 1}, ogf}
```

NOTE. As a by-product, Zeilberger's algorithm delivers a certificate proof for the correctness of the output recurrence:

In[18]:=

```
Prove[]
```

Computer Theorem

Theorem

Let

$$F(k,\ n) = -\frac{\binom{2\,k}{k}\binom{2\,(-k+n-1)}{-k+n-1}}{(k+1)\,(k-n)}$$

and

$$\mathrm{SUM}(n) = \sum_{k=0}^{\infty} F(k,\ n).$$

Then

$$(n+2)\,\mathrm{SUM}(1+n) - 2\,(2\,n+1)\,\mathrm{SUM}(n) = 0.$$

Proof

Let $\Delta_K(".")$ denote the forward difference operator in k and define

$$R(k,\ n) = \frac{(k+1)\,(-2\,k+2\,n-1)\left(4\,k^2 - 2\,k\,(3\,n+2)\right)}{(n^2+n)\,(-k+n+1)}.$$

Then the Theorem follows from summing the equation

$$(n+2)\,F(k,\ 1+n) - 2\,(2\,n+1)\,F(k,\ n) = \Delta_K(F(k,\ n)\,R(k,\ n))$$

over k from 0 to ∞.

Surprise

Instead of Zeilberger's algorithm (1990) already Gosper's algorithm (1978) can solve our problem:

In[19]:=

$$\text{Gosper}\left[\frac{1}{k+1}\binom{2k}{k}_{*}\frac{1}{n-k}\binom{2(n-1-k)}{n-1-k}_{*},\ \{k, 0, n-1\}\right]$$

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If `-1 + n' is a natural number and $n+n^2 \neq 0$, then:

Out[19]=

$$\left\{\text{Sum}\left[-\frac{\text{Binomial}[2\,k,\ k]\ \text{Binomial}[2\,(-1-k+n),\ -1-k+n]}{(1+k)\ (k-n)},\ \{k,\ 0,\ -1+n\}\right] == \frac{2\,(-1+2\,n)\ \text{Binomial}[2\,(-1+n),\ -1+n]}{n+n^2}\right\}$$

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NOTE. The numbers c_n are the celebrated Catalan numbers; e.g., see Richard Stanley's book.

Surprise ⇒ Algorithmic Theory of Contiguous Relations

- In our "Gosper-surprise" Zeilberger's algorithm delivered the minimal possible (homogeneous) recurrence for the sum.
- This is not always the case! ("Zeilberger-surprise")

Surprise $\Rightarrow$ Algorithmic Theory of Contiguous Relations

- In our "Gosper-surprise" Zeilberger's algorithm delivered the minimal possible (homogeneous) recurrence for the sum.
- This is not always the case! ("Zeilberger-surprise")

EXAMPLE (communicated by H. Prodinger). For $m \geq 0$,

$$S(m) := \sum_{k=1}^{2m+1} (-1)^k \binom{2m+1}{k}^2 \binom{2m+1}{k-1} = (-1)^{m+1} \frac{(3m+2)!}{2(m+1)!^2 m!}.$$

But Zeilberger's algorithm does not deliver the minimal recurrence:

In[44]:=

$$\mathrm{Zb}\left[(-1)^k \binom{2m+1}{k}_*^2 \binom{2m+1}{k-1}_*, \{k, 1, 2m+1\}, m, 1\right]$$

Out[44]=

$$\{\}$$

In[45]:=

$$\mathrm{Zb}\left[(-1)^k \binom{2m+1}{k}_*^2 \binom{2m+1}{k-1}_*, \{k, 1, 2m+1\}, m, 2\right]$$

If `2 m' is a natural number, then:

Out[45]=

$$\{-9\,(1+m)\,(4+3m)\,(5+3m)\,(9+4m)\,(5+6m)\,(7+6m)\,\mathrm{SUM}[m] - 6\,(2+m)\,(7+4m)\,\left(345+784m+665m^2+252m^3+36m^4\right)\mathrm{SUM}[1+m] - (2+m)\,(3+m)^2\,(3+2m)\,(5+2m)\,(5+4m)\,\mathrm{SUM}[2+m] == 0\}$$

Now we apply what Zeilberger calls “Paule’s creative symmetrizing”:

$$\text{Rewrite } S(m) = \sum_{k=1}^{2m+1} f(2m+1,k) \text{ as}$$
$$= \sum_{k=1}^{2m+1} \frac{f(2m+1,k) + f(2m+1,2m+1-k)}{2}.$$

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In[52]:=

$$\text{Zb}\left[(-1)^{k}\,\frac{m+1}{2m-k+2}\binom{2m+1}{k}_{*}^{2}\binom{2m+1}{k-1}_{*},\ \{k,1,2m+1\},\ m,\ 1\right]$$

```
If `2 m' is a natural number, then:
```

Out[52]=

$$\{-3\,(4+3\,m)\,(5+3\,m)\,\text{SUM}[m] - (2+m)^{2}\,\text{SUM}[1+m] == 0\}$$

- The order-reduction-effect of "creative symmetrizing" can be explained via an algorithmic theory of contiguous relations.

  Pioneering work: Nobuki Takayama ["Gröbner Basis and the Problem of Contiguous Relations", 1989].

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- Contiguous relations were first studied by Gauß (1813):

$${}_2F_1\left(\begin{matrix} a\pm 1 & b \\ c & \end{matrix};\, x\right),\quad {}_2F_1\left(\begin{matrix} a & b\pm 1 \\ c & \end{matrix};\, x\right),\quad {}_2F_1\left(\begin{matrix} a & b \\ c\pm 1 & \end{matrix};\, x\right)$$

are contiguous to

$${}_2F_1\left(\begin{matrix} a & b \\ c & \end{matrix};\, x\right) := \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k}\, x^k.$$

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are contiguous to

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"There must be many universities to-day where 95 per cent, if not 100 per cent, of the functions studied by physics, engineering, and even mathematics students, are covered by this single symbol ${}_2F_1$."
[W.W. Sawyer, Prelude to Mathematics, Baltimore, Penguin, 1955.]

EXAMPLES.

$$e^{x} = \lim_{a\to\infty} {}_2F_1\left(\begin{matrix} a & 1 \\ 1 & \end{matrix} ; \frac{x}{a}\right);$$

$$\log(1-x) = x\ {}_2F_1\left(\begin{matrix} 1 & 1 \\ 2 & \end{matrix} ; -x\right);$$

$$\sin(x) = x \lim_{a,b\to\infty} {}_2F_1\left(\begin{matrix} a & b \\ \frac{3}{2} & \end{matrix} ; -\frac{x^2/4}{a\,b}\right);$$

$$\cos(x) = \lim_{a,b\to\infty} {}_2F_1\left(\begin{matrix} a & b \\ \frac{1}{2} & \end{matrix} ; -\frac{x^2/4}{a\,b}\right);$$

EXAMPLES contd.

$$\cosh(x) = \lim_{a,b\to\infty} {}_2F_1\left(\begin{matrix} a & b \\ \frac{1}{2} & \end{matrix} ; \frac{x^2/4}{a\,b}\right);$$

$$\arcsin(x) = x\,{}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \end{matrix} ; -x^2\right);$$

$$\arctan(x) = x\,{}_2F_1\left(\begin{matrix} \frac{1}{2} & 1 \\ \frac{3}{2} & \end{matrix} ; -x^2\right);$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2\,x}{\sqrt{\pi}}\lim_{b\to\infty} {}_2F_1\left(\begin{matrix} \frac{1}{2} & b \\ \frac{3}{2} & \end{matrix} ; \frac{x}{b}\right)$$

NOTE. 

- Z's algorithm and the algorithmic theory of contiguous relations carry over to q-hypergeometric functions and q-identities.

For example, one can algorithmically prove polynomial versions [P. '94, P. & Riese '97] of identities like[1]:

$$1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$$

→ "q-Series and Modular Functions"

[1] 1st Rogers-Ramanujan identity

Commercial Break

Commercial Break

Back to Artin's Generating Function

RECALL $F(x) = c_0 x + c_1 x^2 + \cdots + c_{n-1} x^n + \cdots$ and

In[10]:=

```
GuessRE[{1, 1, 2, 5, 14, 42}, c[k]]
```

Out[10]=

```
{{-2 (1 + 2 k) c[k] + (2 + k) c[1 + k] == 0, c[0] == 1}, ogf}
```

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```

In[41]:=

```
RE2DE[{-2 (1 + 2 k) c[k] + (2 + k) c[1 + k] == 0,
       c[0] == 1}, c[k], F[x]]
```

Out[41]=

$$\{-1 - (-1 + 2\,x)\,F[x] - (-x + 4\,x^2)\,F'[x] == 0,\ F[0] == 1\}$$

In[42]:=

```
DSolve[%, F[x], x]
```

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```
RE2DE[{-2 (1 + 2 k) c[k] + (2 + k) c[1 + k] == 0,
       c[0] == 1}, c[k], F[x]]
```

Out[41]=

$$\left\{-1 - (-1 + 2x)\, F[x] - \left(-x + 4x^2\right) F'[x] == 0,\ F[0] == 1\right\}$$

In[42]:=

```
DSolve[%, F[x], x]
```

Out[42]=

$$\left\{\left\{F[x] \to \frac{1 - \sqrt{1 - 4x}}{2x}\right\}\right\}$$

Koutschan's Holonomic Functions Package

EXAMPLE. Background: relativistic Coulomb integrals [Koutschan, P. & Suslov '14]

$$A_p = \int_0^\infty r^{p+2} \left(F(r)^2 + G(r)^2\right) \, dr,$$

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$$A_p = \int_0^\infty r^{p+2} \left(F(r)^2 + G(r)^2\right) dr,$$

where $p \in \mathbb{N}$ and

$$\begin{pmatrix} F(r) \\ G(r) \end{pmatrix} = a^2 \beta^{3/2} \sqrt{\frac{n!}{\gamma\, \Gamma(n+2\nu)}}$$

$$(2 a \beta r)^{\nu-1} \exp(-a \beta r)$$

$$* \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \begin{pmatrix} L_{n-1}^{2\nu}(2 a \beta r) \\ L_n^{2\nu}(2 a \beta r) \end{pmatrix};$$

$n = 0, 1, 2, \ldots$ (radial quantum number),

$$A_p = \int_0^\infty r^{p+2} \left(F(r)^2 + G(r)^2\right) dr,$$

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$n=0,1,2,\ldots$ (radial quantum number),

Laguerre polyomials $L_n^{2\nu}(2 a \beta r)$ given by

$$L_n^\alpha(x) = \frac{\Gamma(\alpha + n + 1)}{n!\, \Gamma(\alpha + 1)}\, {}_1F_1\left(\begin{matrix} -n \\ \alpha + 1 \end{matrix}; x\right).$$

Using non-commutatative GB and holonomic closure properties, Koutschan's package computes a recurrence for the integral A_p:

Using non-commutatative GB and holonomic closure properties, Koutschan's package computes a recurrence for the integral A_p:

```
FindCreativeTelescoping[CpIntegrand, Der[r]]
%[[1, 2]]
```

$$\left(8 a^2 n^2 \beta^2 \alpha_1^2 \beta_1^2 + 4 a^2 n^2 p \beta^2 \alpha_1^2 \beta_1^2 + 8 a^2 \beta^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 16 a^2 n^2 \beta^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 20 a^2 p \beta^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 8 a^2 n^2 p \beta^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 16 a^2 p^2 \beta^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 4 a^2 p^3 \beta^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 32 a^2 n \beta^2 \nu \alpha_1 \alpha_2 \beta_1 \beta_2 + 16 a^2 n p \beta^2 \nu \alpha_1 \alpha_2 \beta_1 \beta_2 + 8 a^2 n^2 \beta^2 \alpha_2^2 \beta_2^2 + 4 a^2 n^2 p \beta^2 \alpha_2^2 \beta_2^2 + 32 a^2 n \beta^2 \nu \alpha_2^2 \beta_2^2 + 16 a^2 n p \beta^2 \nu \alpha_2^2 \beta_2^2 + 32 a^2 \beta^2 \nu^2 \alpha_2^2 \beta_2^2 + 16 a^2 p \beta^2 \nu^2 \alpha_2^2 \beta_2^2\right) S_p^2 +$$

$$\left(6 a n^2 \beta \alpha_1^2 \beta_1^2 - 12 a n^3 \beta \alpha_1^2 \beta_1^2 + 4 a n^2 p \beta \alpha_1^2 \beta_1^2 - 8 a n^3 p \beta \alpha_1^2 \beta_1^2 - 12 a n^2 \beta \nu \alpha_1^2 \beta_1^2 - 8 a n^2 p \beta \nu \alpha_1^2 \beta_1^2 - 24 a n \beta \alpha_1 \alpha_2 \beta_1 \beta_2 - 24 a n^3 \beta \alpha_1 \alpha_2 \beta_1 \beta_2 - 52 a n p \beta \alpha_1 \alpha_2 \beta_1 \beta_2 - 16 a n^3 p \beta \alpha_1 \alpha_2 \beta_1 \beta_2 - 36 a n p^2 \beta \alpha_1 \alpha_2 \beta_1 \beta_2 - 8 a n p^3 \beta \alpha_1 \alpha_2 \beta_1 \beta_2 - 24 a \beta \nu \alpha_1 \alpha_2 \beta_1 \beta_2 - 72 a n^2 \beta \nu \alpha_1 \alpha_2 \beta_1 \beta_2 - 52 a p \beta \nu \alpha_1 \alpha_2 \beta_1 \beta_2 - 48 a n^2 p \beta \nu \alpha_1 \alpha_2 \beta_1 \beta_2 - 36 a p^2 \beta \nu \alpha_1 \alpha_2 \beta_1 \beta_2 - 8 a p^3 \beta \nu \alpha_1 \alpha_2 \beta_1 \beta_2 - 48 a n \beta \nu^2 \alpha_1 \alpha_2 \beta_1 \beta_2 - 32 a n p \beta \nu^2 \alpha_1 \alpha_2 \beta_1 \beta_2 - 6 a n^2 \beta \alpha_2^2 \beta_2^2 - 12 a n^3 \beta \alpha_2^2 \beta_2^2 - 4 a n^2 p \beta \alpha_2^2 \beta_2^2 - 8 a n^3 p \beta \alpha_2^2 \beta_2^2 - 24 a n \beta \nu \alpha_2^2 \beta_2^2 - 60 a n^2 \beta \nu \alpha_2^2 \beta_2^2 - 16 a n p \beta \nu \alpha_2^2 \beta_2^2 - 40 a n^2 p \beta \nu \alpha_2^2 \beta_2^2 - 24 a \beta \nu^2 \alpha_2^2 \beta_2^2 - 96 a n \beta \nu^2 \alpha_2^2 \beta_2^2 - 16 a p \beta \nu^2 \alpha_2^2 \beta_2^2 - 64 a n p \beta \nu^2 \alpha_2^2 \beta_2^2 - 48 a \beta \nu^3 \alpha_2^2 \beta_2^2 - 32 a p \beta \nu^3 \alpha_2^2 \beta_2^2\right) S_p +$$

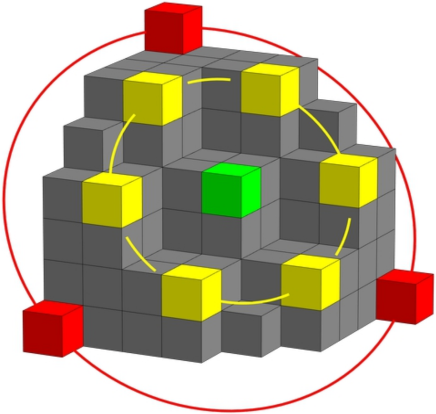
$$\left(-n^2 \alpha_1^2 \beta_1^2 - 3 n^2 p \alpha_1^2 \beta_1^2 - 3 n^2 p^2 \alpha_1^2 \beta_1^2 - n^2 p^3 \alpha_1^2 \beta_1^2 + 4 n^2 \nu^2 \alpha_1^2 \beta_1^2 + 4 n^2 p \nu^2 \alpha_1^2 \beta_1^2 - 4 \alpha_1 \alpha_2 \beta_1 \beta_2 - 2 n^2 \alpha_1 \alpha_2 \beta_1 \beta_2 - 16 p \alpha_1 \alpha_2 \beta_1 \beta_2 - 6 n^2 p \alpha_1 \alpha_2 \beta_1 \beta_2 - 25 p^2 \alpha_1 \alpha_2 \beta_1 \beta_2 - 6 n^2 p^2 \alpha_1 \alpha_2 \beta_1 \beta_2 - 19 p^3 \alpha_1 \alpha_2 \beta_1 \beta_2 - 2 n^2 p^3 \alpha_1 \alpha_2 \beta_1 \beta_2 - 7 p^4 \alpha_1 \alpha_2 \beta_1 \beta_2 - p^5 \alpha_1 \alpha_2 \beta_1 \beta_2 - 4 n \nu \alpha_1 \alpha_2 \beta_1 \beta_2 - 12 n p \nu \alpha_1 \alpha_2 \beta_1 \beta_2 - 12 n p^2 \nu \alpha_1 \alpha_2 \beta_1 \beta_2 - 4 n p^3 \nu \alpha_1 \alpha_2 \beta_1 \beta_2 + 16 \nu^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 8 n^2 \nu^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 32 p \nu^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 8 n^2 p \nu^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 20 p^2 \nu^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 4 p^3 \nu^2 \alpha_1 \alpha_2 \beta_1 \beta_2 + 16 n \nu^3 \alpha_1 \alpha_2 \beta_1 \beta_2 + 16 n p \nu^3 \alpha_1 \alpha_2 \beta_1 \beta_2 - n^2 \alpha_2^2 \beta_2^2 - 3 n^2 p \alpha_2^2 \beta_2^2 - 3 n^2 p^2 \alpha_2^2 \beta_2^2 - n^2 p^3 \alpha_2^2 \beta_2^2 - 4 n \nu \alpha_2^2 \beta_2^2 - 12 n p \nu \alpha_2^2 \beta_2^2 - 12 n p^2 \nu \alpha_2^2 \beta_2^2 - 4 n p^3 \nu \alpha_2^2 \beta_2^2 - 4 \nu^2 \alpha_2^2 \beta_2^2 + 4 n^2 \nu^2 \alpha_2^2 \beta_2^2 - 12 p \nu^2 \alpha_2^2 \beta_2^2 + 4 n^2 p \nu^2 \alpha_2^2 \beta_2^2 - 12 p^2 \nu^2 \alpha_2^2 \beta_2^2 - 4 p^3 \nu^2 \alpha_2^2 \beta_2^2 + 16 n \nu^3 \alpha_2^2 \beta_2^2 + 16 n p \nu^3 \alpha_2^2 \beta_2^2 + 16 \nu^4 \alpha_2^2 \beta_2^2 + 16 p \nu^4 \alpha_2^2 \beta_2^2\right)$$

AMS David P. Robbins Prize 2016

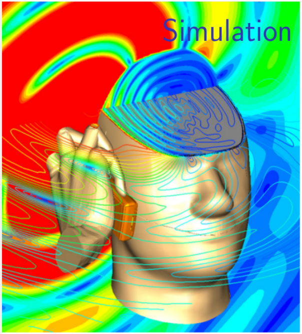
for paper in: Proceedings of the National Academy of Sciences (PNAS) **108(6)**, 2011;

Proof of George Andrews's and David Robbins's q-TSPP conjecture

(by Christoph Koutschan, Manuel Kauers, Doron Zeilberger)



Another Holonomic Surprise: a Patent!



www.cst.com

Simulation of electromagnetic waves

- joint work by Joachim Schöberl (RWTH Aachen), Peter Paule and Christoph Koutschan (RISC)
- wide range of applications in constructing antennas, mobile phones, etc.
- merchandised by the company CST (Computer Simulation Technology)
- simulation with finite element methods
- significant contributions from Symbolic Computation using CK's package `HolonomicFunctions`
- symbolically derived formulae allow a considerable speed-up
- method is planned to be registered as a patent

Mathematical and physical background

Simulate the propagation of electromagnetic waves using the Maxwell equations

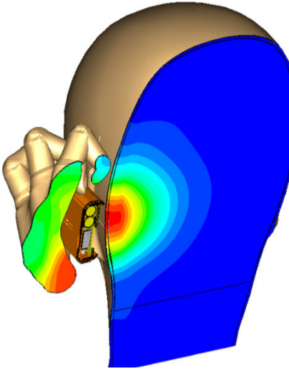
$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \quad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H$$

where H and E are the magnetic and the electric field respectively.

Define basis functions (in 2D) in order to approximate the solution:

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\frac{2y}{1-x} - 1\right)$$

Basis functions in 3D are more involved.



Results

In order to speed up the numerical computations, certain relations for the basis functions $\varphi_{i,j}(x,y)$ are needed.

Using `HolonomicFunctions` relations like the following can easily be derived:

$$
\begin{aligned}
&2(i+2j+5)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)\\
&+(2i+1)(i+2j+1)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)\\
&-(j+3)(i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)\\
&+(j+1)(2i+2j+7)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)\\
&-2(2i+3)(i+j+3)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)\\
&+(i+2j+5)(2i+2j+5)\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y)=\\
&2(i+j+4)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)\\
&+2(i+j+2)(i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)
\end{aligned}
$$

Much bigger formulae in the 3D case! Some efforts were needed to compute them.

Method

- basis functions $\varphi_{i,j}(x,y)$ are composed of functions that are holonomic and ∂-finite, i.e., hypergeometric expressions and orthogonal polynomials (Legendre and Jacobi)
- differential equations and recurrence relations for these objects are known
- symbolic algorithms deliver relations for $\varphi_{i,j}(x,y)$
- compute a Gröbner basis for the ideal of such relations
- search in the ideal for relations of the desired form
- all the above steps can be performed automatically by `HolonomicFunctions`

q-Series and Modular Functions

Current project with Silviu Radu: algebraic relations of modular functions

There are two Rogers-Ramanujan identities:

$$R(q):=1+\sum_{k=1}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

and

$$S(q):=1+\sum_{k=1}^{\infty}\frac{q^{k^2+k}}{(1-q)(1-q^2)\dots(1-q^k)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

Ramanujan (1887-1920) discovered 40 algebraic relations between $R(q)$ and $S(q)$; for example:

$$R(q^{11})S(q)-q^2R(q)S(q^{11})=1.$$

Setting

$$q = q(\tau) := e^{2\pi i \tau},$$

$R(q)$ and $S(q)$ can be considered as modular functions on the upper half of the complex plane.

To my SURPRISE this analytic context can be transferred into a new computer algebra framework [Radu '14, P. & Radu '16].

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- constructive versions of theorems about meromorphic functions on compact Riemann surfaces, and
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NOTE. q-series are strongly related to partitions of numbers.

Example: $p(4) = 5$: $4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$

Example: $p(4) = 5$: $4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$

NOTE. The generating function of the partition numbers is

$$\begin{aligned}
\sum_{n=0}^{\infty} p(n) q^n &= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\
&= (1 + q^1 + q^{1+1} + q^{1+1+1} + \dots) \\
&\times\ (1 + q^2 + q^{2+2} + q^{2+2+2} + \dots) \\
&\times\ \text{etc.} \\
&= \dots + q^{1+1+1} q^{2+2} \dots + \dots
\end{aligned}$$

Let's look at a table of partition numbers from $p(0) := 1$ to $p(80)$:

1	1	2	3	5
7	11	15	22	30
42	56	77	101	135
176	231	297	385	490
627	792	1002	1255	1575
1958	2436	3010	3718	4565
5604	6842	8349	10143	12310
14883	17977	21637	26015	31185
37338	44583	53174	63261	75175
89134	105558	124754	147273	173525
204226	239943	281589	329931	386155
451276	526823	614154	715220	831820
966467	1121505	1300156	1505499	1741630
2012558	2323520	2679689	3087735	3554345
4087968	4697205	5392783	6185689	7089500
8118264	9289091	10619863	12132164	13848650

Ramanujan's Congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

Define: $$\sum_{n=0}^{\infty} p(n) q^n := \prod_{j=1}^{\infty} \frac{1}{1-q^j} :$$

Ramanujan [1919] proved:

$$\sum_{n=0}^{\infty} p(5n+4) q^n = 5 \prod_{j=1}^{\infty} \frac{(1-q^{5j})^5}{(1-q^j)^6}$$

and

$$\sum_{n=0}^{\infty} p(7n+5) q^n$$
$$= 7 \prod_{j=1}^{\infty} \frac{(1-q^{7j})^3}{(1-q^j)^4} + 49q \prod_{j=1}^{\infty} \frac{(1-q^{7j})^7}{(1-q^j)^8}.$$

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What about $p(11n+6)$?

Almost 100 hundred years later Radu's "Ramanujan-Kolberg" package computes in $E^{\infty}(22)$:

$$\sum_{n=0}^{\infty} p(11n+6)q^n = q^{14}\prod_{j=1}^{\infty}\frac{(1-q^{22j})^{22}}{(1-q^j)^{10}(1-q^{2j})^2(1-q^{11j})^{11}}$$
$$\times(1078t^4+13893t^3+31647t^2+11209t-21967$$
$$+z_1(187t^3+5390t^2+594t-9581)$$
$$+z_2(11t^3+2761t^2+5368t-6754)$$

with

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with

$$t := \frac{3}{88}w_1+\frac{1}{11}w_2-\frac{1}{8}w_3,\ z_1 := -\frac{5}{88}w_1+\frac{2}{11}w_2-\frac{1}{8}w_3-3,$$
$$z_2 := \frac{1}{44}w_1-\frac{3}{11}w_2+\frac{5}{4}w_3,$$

where the $w_j \in E^\infty(22)$ are of the form

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$$z_2{:=}\frac{1}{44}w_1-\frac{3}{11}w_2+\frac{5}{4}w_3,$$

where the $w_j \in E^{\infty}(22)$ are of the form

$$w_j = \prod_{j=1}^{\infty}\frac{(1-q^{\square j})^{\square}\dots}{(1-q^{\square j})^{\square}\dots}.$$

This implies that the q-series t, w_1, and w_2 have coefficients in $\mathbb{Z}$. $\square$

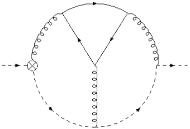
Symbolic Summation in QFT

Symbolic Summation in Quantum Field Theory

JKU Collaboration with DESY (Berlin–Zeuthen)
(Deutsches Elektronen–Synchrotron)

Project leader:	Carsten Schneider (RISC)
Partners:	Johannes Blümlein (DESY)
	Peter Paule (RISC)

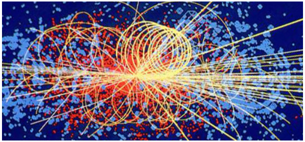
Evaluation of Feynman diagrams



Behavior of particles

$$\longrightarrow \int \Phi(N,\epsilon,x)dx$$

Feynman integrals



Evaluations required for the LHC experiment at CERN

DESY

$$\sum f(N,\epsilon,k)$$

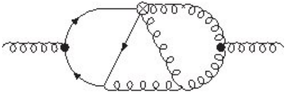
multi-sums

RISC

(symbolic summation)

simple sum expressions

processable by physicists



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$\|$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$

$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$

$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$

$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$

$$+2S_1(s-1)-2S_1(r+s)\Big] + \textbf{3 further 6–fold sums}$$

$\boxed{F_0(N)}=$ (using `Sigma.m`, `EvaluateMultiSums.m` and J. Ablinger's `HarmonicSums.m` package)

$$\frac{7}{12}S_1(N)^4+\frac{(17N+5)S_1(N)^3}{3N(N+1)}+\Big(\frac{35N^2-2N-5}{2N^2(N+1)^2}+\frac{13S_2(N)}{2}+\frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2}+\Big(\frac{4(-1)^N(2N+1)}{N(N+1)}-\frac{13}{N}\Big)S_2(N)+\Big(\frac{29}{3}-(-1)^N\Big)S_3(N)$$
$$+\big(2+2(-1)^N\big)S_{2,1}(N)-28S_{-2,1}(N)+\frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N)+\Big(\frac{3}{4}+(-1)^N\Big)S_2(N)^2$$
$$-2(-1)^NS_{-2}(N)^2+S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)}+\big(26+4(-1)^N\big)S_1(N)+\frac{4(-1)^N}{N+1}\Big)$$
$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)}-\frac{5}{2N^2}\Big)S_2(N)+S_{-2}(N)\Big(10S_1(N)^2+\Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N)+\frac{8(-1)^N(3N+1)}{N(N+1)^2}+\big(-22+6(-1)^N\big)S_2(N)-\frac{16}{N(N+1)}\Big)$$
$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)}-\frac{29}{3N}\Big)S_3(N)+\Big(\frac{19}{2}-2(-1)^N\Big)S_4(N)+\big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)}-\frac{2}{N}\Big)S_{2,1}(N)+\big(20+2(-1)^N\big)S_{2,-2}(N)+\big(-17+13(-1)^N\big)S_{3,}$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N)-\big(24+4(-1)^N\big)S_{-3,1}(N)+\big(3-5(-1)^N\big)S_{2,1}$$
$$+32S_{-2,1,1}(N)+\left(\frac{3}{2}S_1(N)^2-\frac{3S_1(N)}{N}+\frac{3}{2}(-1)^NS_{-2}(N)\right)\zeta(2)$$

$\boxed{F_0(N)} =$ (using `Sigma.m`, `EvaluateMultiSums.m` and J. Ablinger's `HarmonicSums.m` package)

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$-2(-1) \dots -1)^N\big)S_1(N) + \frac{4(-1)^N}{\dots}\Big)$$

$$+\Big(\frac{(-1) \dots}{2N \dots} \dots N)^2 + \Big(\frac{8(-1)^N}{N(\dots}$$

$$+\frac{4(3N \dots}{N(N \dots} \dots 6(-1)^N\big)S_2(N) - \frac{\dots}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N \dots}{N(N+1)} - \frac{\dots}{3N}\Big)S_3(N) + \Big(\frac{\dots}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,\dots}$$

$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,\dots}$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i^2}$$

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Challenges of the project

$\boxed{\text{About 1000 difficult Feynman diagrams have been treated so far}}$

(some took 50 days of calculation time)

$\downarrow$

$\boxed{\text{About a million multi-sums have been simplified}}$

(most were double and triple sums)

Resources

- up to 9 full time employed researchers at RISC/DESY
- 4 up-to-date mainframe DESY computers at RISC
  + exploiting DESY's computer farms
- New computer algebra/special functions technologies
  (new/tuned algorithms, efficient implementations,...)

The STAM Project

The STAM Project

$\rightarrow$ talk by Bruno Buchberger (Wednesday, 11 a.m.)

Conclusion

Andrews ["q-SERIES", 1986] about Ramanujan:

"Sometimes when studying his work I have wondered how much Ramanujan could have done if he had had MACSYMA or SCRATCHPAD or some other symbolic algebra package.

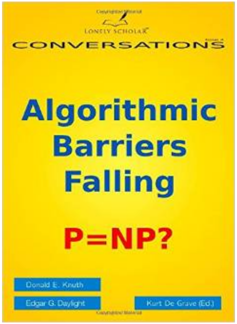
 

Andrews ["q-SERIES", 1986] about Ramanujan:

"Sometimes when studying his work I have wondered how much Ramanujan could have done if he had had MACSYMA or SCRATCHPAD or some other symbolic algebra package. More often I get the feeling that he was such a brilliant, clever, and intuitive computer himself that he really did not need them."

But let's conclude with Knuth:

Recall:



Knuth: Learning how to manipulate formulas fluently, and how to see patterns in formulas instead of patterns in numbers — that's what my book "Concrete Mathematics" is essentially about.

SYMBOLIC COMPUTATION and SOFTWARE not only greatly assist in these tasks, but also can be used to enhance mathematical theory!