Homework 9.11.18

Exercise 1 *Show that the Pochhammer polynomials defined by* $(x)_0 = 1$ *and*

$$(x)_n = \prod_{k=0}^{n-1} (x+k), \quad n \in \mathbb{N},$$

satisfy the identities

(i)

$$(x)_{n+m} = (x)_n (x+n)_m, \quad n, m \in \mathbb{N}_0,$$

(ii)

$$(-x)_n = (-1)^n (x-n+1)_n,$$

and

(iii)

$$\frac{(x-m)_n}{(x)_n} = \frac{(x-m)_m}{(x-m+n)_m} = \frac{(1-x)_m}{(1-x-n)_m}, \quad n, m \in \mathbb{N}_0.$$

Exercise 2 *Show that the falling factorial polynomials defined by* $\phi_0(x) = 1$ *and*

$$\phi_n(x) = \prod_{j=0}^{n-1} (x-j), \quad n \in \mathbb{N},$$

satisfy the identities

(i)

$$\phi_n(x) = (-1)^n (-x)_n = (x-n+1)_n,$$

(ii)

$$\phi_n(x+1) = \phi_n(x) + n\phi_{n-1}(x),$$

(iii)

$$\Delta^i \phi_j(x) = \phi_i(j)\, \phi_{j-i}(x),$$

where Δ *is the forward difference operator.*

Exercise 3 *Let the moments* $\nu_n(z)$ *be defined by*

$$\nu_n(z) = \sum_{x=0}^{\infty} \phi_n(x) \frac{z^x}{x!}.$$

Show that

(i)

$$\nu_0(z) = e^z.$$

(ii)

$$\nu_{n+1} - z\nu_n = 0.$$

(iii)

$$\nu_n(z) = z^n e^z.$$

Exercise 4 *Let the entries of the Gram matrix G be defined by*

$$G_{i,j} = e^z \sum_{k=0}^{\infty} \binom{i}{k} \binom{j}{k} k! z^{i+j-k}.$$

Show that

$$G_{i,j} = e^z z^{i+j} \; {}_2F_0 \left(\begin{matrix} -i, -j \\ - \end{matrix} ; z^{-1} \right).$$

Exercise 5 *Let*

$$C_{i,j} = \binom{i}{j} z^{i-j}, \quad H_{i,j} = \; i! \; z^i e^z \delta_{i,j}, \quad i, j \in \mathbb{N}_0.$$

Show that

(i)

$$G = CHC^T,$$

where G is the matrix defined in the previous exercise.

(ii) If

$$A_{i,j} = \sum_{k=0}^{\infty} (-1)^{i-k} C_{i,k} C_{k,j}, \quad i, j \in \mathbb{N}_0,$$

then

$$\begin{aligned} A_{i,j} &= 0, \quad i < j \\ A_{i,j} &= 1, \quad i = j. \end{aligned}$$