Matrix factorizations and orthogonal polynomials

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1 Introduction

Let $\mathbb{N}_0$ denote the set

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = 0, 1, 2, \ldots,$$

and L be a linear functional acting on the space of polynomials $\mathbb{C}[x]$, i.e., belonging to the dual vector space $\mathbb{C}^*[x]$:

$$L[ap + bq] = aL[p] + bL[q], \quad a, b \in \mathbb{C}, \quad p, q \in \mathbb{C}[x].$$

The numbers

$$\mu_n = L[x^n] \in \mathbb{C}, \quad n \in \mathbb{N}_0,$$

are called the **moments** of L.

If we have a sequence of polynomials $p_n(x) \in \mathbb{C}[x]$,

$$\deg(p_n) = n, \quad n \in \mathbb{N}_0,$$

satisfying

$$\begin{aligned} L\left[x^k p_n\right] &= 0, \quad 0 \leq k < n, \\ L[x^n p_n] &= h_n \neq 0, \end{aligned} \tag{1}$$

for all $n \in \mathbb{N}_0$, we say that $\{p_n\}$ is a family of **orthogonal polynomials** with respect to L. Examples include Legendre, Chebyshev, Jacobi, Hermite, Gegenbauer, Laguerre, Charlier, Kravchuk, Meixner, Hahn polynomials, and many others!

If $n \in \mathbb{N}$ and

$$p_n(x) = \sum_{j=0}^{n} c_j x^j,$$

we have

$$L\left[x^k p_n\right] = \sum_{j=0}^{n} c_j L\left[x^{k+j}\right] = \sum_{j=0}^{n} \mu_{k+j} c_j,$$

and using (1), we get

$$\sum_{j=0}^{n} \mu_{k+j} c_j = 0, \quad 0 \leq k < n,$$

$$\sum_{j=0}^{n} \mu_{n+j} c_j = h_n \neq 0.$$

If we introduce the **Hankel matrix**

$$M_{i,j} = \mu_{i+j}, \quad 0 \leq i, j \leq n,$$

we can write

$$M \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ h_n \end{bmatrix},$$

and we will have a unique solution if the **Hankel determinants** Δ_n satisfy

$$\Delta_n = \det_{0 \leq i,j \leq n} (\mu_{i+j}) \neq 0, \quad n \in \mathbb{N}_0.$$

We say that L is a **quasi-definite functional** if $\Delta_n \neq 0, \quad n \in \mathbb{N}_0$, and L is a **positive-definite functional** if $\Delta_n > 0, \quad n \in \mathbb{N}_0$.

2 Main theory

2.1 Definitions

We begin with a few definitions.

Definition 1 *A* ***semi-infinite matrix*** $A \in \mathbb{C}^{\infty \times \infty}$ *is a function* $A : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$*. We write*

$$A(i,j) = A_{i,j}.$$

(i) We say that A is an ***upper triangular matrix*** *if*

$$A_{i,j} = 0, \quad i > j.$$

We say that U is a ***unit upper triangular*** *(**UUT**) matrix if A is upper triangular and*

$$A_{i,i} = 1, \quad i \in \mathbb{N}_0.$$

In other words,

$$A = \begin{pmatrix} 1 & A_{0,1} & A_{0,2} \\ 0 & 1 & A_{1,2} \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) We say that A is a ***lower triangular matrix*** *if*

$$A_{i,j} = 0, \quad i < j.$$

We say that A is a ***unit lower triangular*** *(**ULT**) matrix if L is lower triangular and*

$$A_{i,i} = 1, \quad i \in \mathbb{N}_0.$$

In other words,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ A_{1,0} & 1 & 0 \\ A_{2,0} & A_{2,1} & 1 \end{pmatrix}.$$

Definition 2 *We say that* $\overrightarrow{q} \in \mathbb{C}[x]^{\infty \times 1}$ *is a* ***basis*** *of* $\mathbb{C}[x]$ *if* $q_n(x) \in \mathbb{C}[x]$ *and* $\deg(q_n) = n$*.*

We say that $\overrightarrow{q}$ *is a* ***monic basis*** *if* $q_n(x)$ *is a monic polynomial for all* $n \in \mathbb{N}_0$*,*

$$q_n(x) = x^n + \cdots.$$

The basis that we will use in our examples is constructed with the falling factorials.

Example 3 *The basis of* ***falling factorial*** *(or binomial) polynomials is defined by* $\phi_0(x) = 1$ *and*

$$\phi_n(x) = \prod_{j=0}^{n-1} (x - j), \quad n \in \mathbb{N}. \tag{2}$$

Using the definition (2), we immediately obtain the recurrence relation

$$\phi_{n+1}(x) = (x-n)\,\phi_n(x). \tag{3}$$

Definition 4 *We define the* ***Pochhammer*** *(or rising factorial) polynomials by* $(x)_0 = 1$ *and*

$$(x)_n = \prod_{k=0}^{n-1}(x+k), \quad n \in \mathbb{N}. \tag{4}$$

Remark 5 *The Pochhammer polynomials can be generalized to complex values of* n *using the formula [6, 5.2.5]*

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad -(x+n) \notin \mathbb{N}_0, \tag{5}$$

where $\Gamma(z)$ *is the Gamma function.*

The Pochhammer polynomials satisfy many identities (HW #1), including the recurrence [5, 18:5:12]

$$(x)_{n+m} = (x)_n (x+n)_m, \quad n, m \in \mathbb{N}_0, \tag{6}$$

the change of sign identity

$$(-x)_n = (-1)^n (x-n+1)_n, \tag{7}$$

and the ratio formulas [5, 18:5:10]

$$\frac{(x-m)_n}{(x)_n} = \frac{(x-m)_m}{(x-m+n)_m} = \frac{(1-x)_m}{(1-x-n)_m}, \quad n, m \in \mathbb{N}_0. \tag{8}$$

We see from (2), (4), and (7) that the polynomials $\phi_n(x)$ and $(x)_n$ are related by (HW #2)

$$\phi_n(x) = (-1)^n (-x)_n = (x-n+1)_n. \tag{9}$$

Note that from (5) and (9), we get

$$\phi_n(x) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)} = n!\binom{x}{n}. \tag{10}$$

The falling factorial polynomials are eigenvalues of the **forward difference operator (**acting on the variable x) defined by

$$\Delta f(x) = f(x+1) - f(x).$$

In fact, we have the following result.

Lemma 6 *For all $i, j \in \mathbb{N}_0$, we have*

$$\Delta^i \phi_j(x) = \phi_i(j)\, \phi_{j-i}(x). \tag{11}$$

Note that from (2) we see that

$$\phi_i(j) = 0, \quad i > j,$$

and therefore

$$\Delta^i \phi_j(x) = 0, \quad i > j.$$

Proof. HW #2. ■

Using the Pochhammer polynomials we can construct the generalized hypergeometric function.

Definition 7 *The* ***generalized hypergeometric function*** ${}_pF_q$ *is defined by [6, 16.2]*

$${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{12}$$

Remark 8 *The convergence of the series (12) depends on the values of p and q. We have three different cases to consider:*

1. If $p < q + 1$, ${}_pF_q$ is an entire function of z.
2. If $p = q + 1$, ${}_pF_q$ is analytic inside the unit circle, $|z| < 1$.
3. If $p > q + 1$, ${}_pF_q$ diverges for $z \neq 0$, unless one or more of the top parameters a_i is a negative integer. If we take $a_1 = -N$, with $N \in \mathbb{N}_0$, then ${}_pF_q$ becomes a polynomial of degree N.

For example, we can write the exponential generating function of the Pochhammer polynomials as a ${}_1F_0$ function.

Example 9 *Using the binomial theorem and (10), we have*

$$(1+z)^x = \sum_{n=0}^{\infty} \binom{x}{n} z^n = \sum_{n=0}^{\infty} \frac{\phi_n(x)}{n!} z^n, \quad |z| < 1.$$

From (9), we get

$${}_1F_0\left(\begin{array}{c} x \\ - \end{array}; z\right) = \sum_{n=0}^{\infty} (x)_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \phi_n(-x) \frac{z^n}{n!} = (1-z)^{-x}, \quad |z| < 1. \tag{13}$$

In the next section, we will need the following result.

Proposition 10 *The polynomials $\phi_n(x)$ satisfy the connection formula*

$$\phi_n(x)\,\phi_m(x) = \sum_{k=0}^{\infty} \binom{n}{k}\binom{m}{k} k!\phi_{n+m-k}(x)\,. \tag{14}$$

Proof. Can you find one? Maybe using symbolic computation? ■

2.2 Linear functionals

Definition 11 *Let $L : \mathbb{C}[x] \to \mathbb{C}$ be a linear functional and $\overrightarrow{q} \in \mathbb{C}[x]^{\infty\times 1}$ be a monic basis.*

(i) The numbers

$$\nu_n = L[q_n]\,, \quad n \in \mathbb{N}_0,$$

are called the ***(generalized) moments*** *of L. We write*

$$\overrightarrow{\nu} = L[\overrightarrow{q}] \in \mathbb{C}[x]^{\infty\times 1}\,.$$

(ii) We define the ***Gram matrix*** *G by*

$$G = L\left[\overrightarrow{q}\ \overrightarrow{q}^T\right] \in \mathbb{C}^{\infty\times\infty}.$$

As an example, we consider the following linear functional.

Example 12 *Let $L : \mathbb{C}[x] \to \mathbb{C}$ be defined by*

$$L[q] = \sum_{x=0}^{\infty} q(x)\frac{z^x}{x!}, \quad q \in \mathbb{C}[x]\,. \tag{15}$$

The moments of L on the falling factorial basis are given by

$$\nu_n(z) = L[\phi_n] = \sum_{x=0}^{\infty} \phi_n(x)\,\frac{z^x}{x!}.$$

We can show (HW #3) that

$$\nu_n(z) = z^n e^z. \tag{16}$$

Using (16) and (14), we obtain (HW #4)

$$G_{i,j} = L[\phi_i, \phi_j] = e^z z^{i+j}\ {}_2F_0\left(\begin{matrix} -i, -j \\ - \end{matrix} ; z^{-1} \right). \tag{17}$$

Remark 13 *Note that the matrix G defined by (17) is symmetric, and all the entries are finite sums, since the hypergeometric series terminates for all $i, j \in \mathbb{N}_0$. Also, $z = 0$ is not a singularity of $G_{i,j}$, since the power z^{i+j} cancels the powers of z^{-1}.*

Definition 14 *We say that L is a* ***quasi-definite functional*** *with respect to a monic basis $\overrightarrow{q} \in \mathbb{C}[x]^{\infty\times 1}$ if the matrix $L\left[\overrightarrow{q}\ \overrightarrow{q}^T\right]$ admits the LDL decomposition [3, 4.12]*

$$L\left[\overrightarrow{q}\ \overrightarrow{q}^T\right] = G = CHC^T, \tag{18}$$

where $C \in \mathbb{C}^{\infty\times\infty}$ is a ULT matrix and $H \in \mathbb{C}^{\infty\times\infty}$ is a nonsingular diagonal matrix

$$H_{i,j} = h_i \delta_{i,j}, \quad h_i \neq 0, \quad i, j \in \mathbb{N}_0.$$

If $h_i > 0$ for all $i \in \mathbb{N}_0$, we say that L is a ***positive-definite functional****.*

Proposition 15 *If L is a quasi-definite functional with respect to $\overrightarrow{q}$, then we can compute the entries of C and H in (18) by the following iterative formula:*

$$\begin{aligned} &h_0 = G_{0,0}, \quad C_{i,0} = \frac{G_{i,0}}{h_0}, \quad C_{i,i} = 1, \quad i \in \mathbb{N}_0, \\ &C_{i,j} = 0, \quad i < j, \end{aligned}$$

and for $i \in \mathbb{N}$,

$$\begin{aligned} C_{i,j} &= \frac{1}{h_j}\left(G_{i,j} - \sum_{k=0}^{j-1} C_{i,k} C_{j,k} h_k\right), \quad j = 1, \ldots, i-1, \\ h_i &= G_{i,i} - \sum_{k=0}^{i-1} \left(C_{i,k}\right)^2 h_k. \end{aligned} \tag{19}$$

Proof. Let $i \geq j$. Then, since C is a ULT matrix we have $C_{j,k} = 0, \quad j < k$, and

$$\begin{aligned} G_{i,j} &= \left(CHC^T\right)_{i,j} = \sum_{k=0}^{\infty} C_{i,k} h_k C_{j,k} \\ &= \sum_{k=0}^{j} C_{i,k} h_k C_{j,k} = C_{i,j} h_j + \sum_{k=0}^{j-1} C_{i,k} h_k C_{j,k}. \end{aligned}$$

Solving for $C_{i,j}$, we get

$$C_{i,j} = \frac{1}{h_j}\left(G_{i,j} - \sum_{k=0}^{j-1} C_{i,k} h_k C_{j,k}\right).$$

In particular, when $i = j$

$$1 = C_{i,i} = \frac{1}{h_i}\left[G_{i,i} - \sum_{k=0}^{i-1} \left(C_{i,k}\right)^2 h_k\right].$$

■

Example 16 *Let the matrix G be defined by (17). Since*

$$h_0 = G_{0,0} = e^z, \quad C_{i,0} = \frac{G_{i,0}}{G_{0,0}} = z^i,$$

we can use (19), and obtain

$$h_1 = ze^z, \quad h_2 = 2z^2e^z, \quad h_3 = 6z^3e^z, \ldots,$$

and

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ z & 1 & 0 & 0 & 0 \\ z^2 & 2z & 1 & 0 & 0 \\ z^3 & 3z^2 & 3z & 1 & 0 \\ z^4 & 4z^3 & 6z^2 & 4z & 1 \end{pmatrix}.$$

We see that the matrices C and H in the LDL decomposition (18) have entries (HW #5)

$$C_{i,j} = \binom{i}{j} z^{i-j}, \quad i, j \in \mathbb{N}_0, \tag{20}$$

and $H_{i,j} = h_i \delta_{i,j}$, with

$$h_i = \ i! \ z^i e^z, \quad i \in \mathbb{N}_0. \tag{21}$$

We conclude that L is a quasi-definite functional if $z \neq 0$. The functional L will be positive definite if $z > 0$.

2.3 Orthogonal polynomials

In this section, we introduce sequences of polynomials orthogonal with respect to linear functionals.

Definition 17 *If L is a quasi-definite functional with respect to $\overrightarrow{q}$, we define the sequence of* ***monic orthogonal polynomials*** *(**MOPS**) with respect to L by*

$$\overrightarrow{p} = C^{-1}\overrightarrow{q} \in \mathbb{C}[x]^{\infty\times 1}. \tag{22}$$

Example 18 *Let the matrix C be defined by (20). Let*

$$\begin{aligned} A_{i,j} &= \sum_{k=0}^{\infty} (-1)^{i-k} C_{i,k}C_{k,j} = \sum_{k=0}^{\infty} (-1)^{i-k} \binom{i}{k} z^{i-k} \binom{k}{j} z^{k-j} \\ &= z^{i-j} \sum_{k=0}^{\infty} (-1)^{i-k} \binom{i}{k}\binom{k}{j}, \quad i,j \in \mathbb{N}_0. \end{aligned}$$

Then (HW #5),

$$\begin{aligned} A_{i,j} &= 0, \quad i < j \\ A_{i,j} &= 1, \quad i = j. \end{aligned}$$

If $i > j$, we get

$$A_{i,j} = z^{i-j} \sum_{k=j}^{i} (-1)^{i-k} \binom{i}{k}\binom{k}{j},$$

and using (10)

$$A_{i,j} = z^{i-j} \sum_{k=j}^{i} (-1)^{i-k} \binom{i}{k} \frac{\phi_j(k)}{j!}.$$

If we use the formula for higher order differences [7, 6.1]

$$\Delta^p f(x) = \sum_{j=0}^{p} \binom{p}{j} (-1)^{p-j} f(x+j), \tag{23}$$

we see that

$$A_{i,j} = \frac{z^{i-j}}{j!} \left[\Delta^i \phi_j(x)\right]_{x=0}, \quad i > j.$$

But since $\phi_j(x)$ is a polynomial of degree j, and $i > j$

$$A_{i,j} = 0, \quad i > j.$$

We conclude that

$$\sum_{k=0}^{\infty} (-1)^{i-k} C_{i,k} C_{k,j} = A_{i,j} = \delta_{i,j}, \quad i, j \in \mathbb{N}_0,$$

and therefore

$$\left(C^{-1}\right)_{i,k} = (-1)^{i-k} C_{i,k}. \tag{24}$$

The polynomials $\overrightarrow{p} = C^{-1}\overrightarrow{\phi}$ are known as (monic) **Charlier polynomials** [4, 6.1]. Using (20) and (24), we get

$$p_n(x) = \sum_{j=0}^{\infty} \left(C^{-1}\right)_{n,j} \phi_j(x) = \sum_{j=0}^{\infty} (-1)^{n-j} \binom{n}{j} z^{n-j} \phi_j(x).$$

From (10) and (9), we have

$$\binom{n}{j} = \frac{\phi_j(n)}{j!} = \frac{(-1)^j (-n)_j}{j!}.$$

Therefore,

$$(-z)^{-j} \binom{n}{j} \phi_j(x) = (-1)^j \binom{n}{j} j! (-1)^j \phi_j(x) \frac{(-z)^{-j}}{j!} = (-n)_j (-x)_j \frac{(-z)^{-j}}{j!},$$

and we obtain the hypergeometric representation [1]

$$p_n(x) = (-z)^n \ {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} ; -z^{-1} \right).$$

Theorem 19 *Let L be a quasi-definite functional with respect to $\overrightarrow{q}$ and $\overrightarrow{p}$ be the corresponding MOPS. Then,*

(i) The polynomials $p_n(x)$ satisfy the ***orthogonality relation***

$$L\left[\overrightarrow{p}\ \overrightarrow{p}^T\right] = H. \tag{25}$$

(ii) We have

$$L\left[\overrightarrow{p}\right] = h_0 \overrightarrow{e_0}, \tag{26}$$

where

$$(\overrightarrow{e_k})_j = \delta_{k,j}.$$

(iii) If $\overrightarrow{\psi}$ *is a monic basis of* $\mathbb{C}[x]$, *then*

$$L\left[\overrightarrow{p}\ \overrightarrow{\psi}^T\right] = HU,$$

where U *is a UUT matrix. In other words, for all* $i, j \in \mathbb{N}_0$

$$L[p_i \psi_j] = \begin{cases} h_i, & i = j \\ 0, & i > j \end{cases}. \tag{27}$$

Proof. (i) Using (22), we have

$$L\left[\overrightarrow{p}\ \overrightarrow{p}^T\right] = L\left[C^{-1}\overrightarrow{q}\ \overrightarrow{q}^T C^{-T}\right] = C^{-1}GC^{-T} = H,$$

where

$$C^{-T} = \left(C^T\right)^{-1} = \left(C^{-1}\right)^T.$$

(ii) Using (25), we have

$$\left(L\left[\overrightarrow{p}\right]\right)_j = L[p_j] = L[p_j p_0] = h_0 \delta_{j,0}.$$

(iii) If $\overrightarrow{\psi}$ is a monic basis of $\mathbb{C}[x]$, then there exists a ULT matrix A such that

$$\overrightarrow{\psi} = A\ \overrightarrow{q}.$$

Using (22), we get

$$L\left[\overrightarrow{p}\ \overrightarrow{\psi}^T\right] = L\left[C^{-1}\overrightarrow{q}\ \overrightarrow{q}^T A^T\right] = C^{-1}GA^T = HC^T A^T.$$

Since C and A are ULT matrices, the matrix $C^T A^T$ is UUT. ■

Example 20 *Meixner polynomials. Using (15), (21) and (25), we obtain the orthogonality relation for the (monic) Meixner polynomials [1]*

$$\sum_{x=0}^{\infty} p_n(x)\, p_m(x)\, (a)_x \frac{z^x}{x!} = e^z n!\ z^n\ \delta_{n,m}, \quad n, m \in \mathbb{N}_0.$$

Definition 21 *Let* $\overrightarrow{p}$ *be the MOPS with respect to a quasi-definite functional* L*. We define the* ***Jacobi matrix*** $J \in \mathbb{C}^{\infty\times\infty}$ *by*

$$J = L\left[x\overrightarrow{p}\ \overrightarrow{p}^T\right] H^{-1}. \tag{28}$$

Theorem 22 *(i) The Jacobi matrix* J *defined by (28) is a tridiagonal matrix with entries*

$$J_{i,j} = \delta_{i+1,j} + \beta_i\delta_{i,j} + \gamma_i\delta_{i-1,j}, \tag{29}$$

where the coefficients β_i, γ_i *are given by*

$$\beta_i = \frac{L\left[xp_i^2\right]}{h_i}, \quad i \in \mathbb{N}_0,$$

$\gamma_0 = 0$ *and*

$$\gamma_i = \frac{L\left[xp_ip_{i-1}\right]}{h_{i-1}} = \frac{h_i}{h_{i-1}} \neq 0, \quad i \in \mathbb{N}. \tag{30}$$

(ii) The polynomials $\overrightarrow{p}$ *satisfy the eigenvalue equation*

$$J\ \overrightarrow{p} = x\overrightarrow{p}. \tag{31}$$

By linearity, this extends to

$$q(x)\overrightarrow{p} = q(J)\overrightarrow{p}, \quad q \in \mathbb{C}[x]. \tag{32}$$

(iii) Let $q \in \mathbb{C}[x]$*. Then,* $q(J)H$ *is a symmetric matrix.*
(iv) Let $q \in \mathbb{C}[x]$ *be given by*

$$q(x) = \ \overrightarrow{p}^T\ \overrightarrow{\omega}, \quad \overrightarrow{\omega} \in \mathbb{C}[x]^{\infty\times 1}. \tag{33}$$

Then,

$$\omega_k = \frac{h_0}{h_k}\left[q(J)\right]_{k,0}. \tag{34}$$

Proof. (i) Using (27) in two different ways, we have

$$L\left[p_i\ xp_j\right] = \begin{cases} h_i, & i = j+1 \\ 0, & i > j+1 \end{cases},$$

and

$$L\left[p_j\ xp_i\right] = \begin{cases} h_j, & j = i+1 \\ 0, & j > i+1 \end{cases}.$$

Thus, from (28) we obtain

$$(JH)_{i,j} = 0, \quad j \notin \{i-1, i, i+1\}.$$

The three nonzero entries are given by

$$J_{i,i-1}h_{i-1} = L[xp_i p_{i-1}] = h_i,$$
$$J_{i,i}h_i = L[xp_i^2] = h_i\beta_i,$$

and

$$J_{i,i+1}h_{i+1} = L[xp_i p_{i+1}] = h_{i+1}.$$

(ii) Representing $x\overrightarrow{p}$ with respect to the basis $\overrightarrow{p}$, we have

$$x\overrightarrow{p} = M\overrightarrow{p},$$

for some matrix M. Multiplying by $\overrightarrow{p}^T$ and applying L on both sides of the equation, we get

$$JH = L\left[x\overrightarrow{p}\ \overrightarrow{p}^T\right] = ML\left[\overrightarrow{p}\ \overrightarrow{p}^T\right] = MH,$$

where we have used (25) and (28). Since H is nonsingular, $M = J$.

(iii) Using (32), we have

$$L\left[q\overrightarrow{p}\ \overrightarrow{p}^T\right] = L\left[q(J)\overrightarrow{p}\ \overrightarrow{p}^T\right] = q(J)L\left[\overrightarrow{p}\ \overrightarrow{p}^T\right] = q(J)H.$$

But on the other hand,

$$L\left[q\overrightarrow{p}\ \overrightarrow{p}^T\right] = L\left[\overrightarrow{p}\ \overrightarrow{p}^T q\right] = L\left[\overrightarrow{p}\ \overrightarrow{p}^T q\left(J^T\right)\right] = Hq\left(J^T\right).$$

Therefore,

$$[q(J)H]^T = H^T[q(J)]^T = Hq\left(J^T\right) = q(J)H. \tag{35}$$

(iv) From (33), we have

$$L[\overrightarrow{p}q] = L\left[\overrightarrow{p}\ \overrightarrow{p}^T\ \overrightarrow{\omega}\right] = H\ \overrightarrow{\omega}.$$

Using (32),

$$L[\overrightarrow{p}q] = L[q\overrightarrow{p}] = L[q(J)\overrightarrow{p}] = q(J)L[\overrightarrow{p}].$$

Finally, from (26)

$$q(J)L[\overrightarrow{p}] = q(J)h_0\overrightarrow{e_0}.$$

Thus, we conclude that

$$h_j\omega_j = (H\ \overrightarrow{\omega})_j = \sum_k [q(J)]_{j,k} h_0\delta_{k,0} = h_0[q(J)]_{j,0}.$$

■

Corollary 23 *Let $\overrightarrow{p}$ be the MOPS with respect to a quasi-definite functional L. Then, the polynomials $\overrightarrow{p}$ satisfy the* ***three-term recurrence relation***

$$xp_n = p_{n+1} + \beta_n p_n + \gamma_n p_{n-1}, \quad n \in \mathbb{N}_0, \tag{36}$$

with initial conditions

$$p_{-1} = 0, \quad p_0 = 1.$$

The following result is known as the Modified Chebyshev algorithm [2, 2.1.7].

Proposition 24 *Let $\overrightarrow{p}$ be the MOPS with respect to a quasi-definite functional L and $\overrightarrow{q}$ be a monic basis of $\mathbb{C}[x]$ satisfying*

$$x\overrightarrow{q} = T\ \overrightarrow{q}, \tag{37}$$

where T is a tridiagonal matrix with entries

$$T_{i,j} = \delta_{i+1,j} + \eta_i \delta_{i,j} + \xi_i \delta_{i-1,j}. \tag{38}$$

Let the "modified moments" be defined by

$$R = L\left[\overrightarrow{q}\ \overrightarrow{p}^T\right].$$

Then, the entries of R satisfy the recurrence

$$R_{i,j+1} = R_{i+1,j} + (\eta_i - \beta_j) R_{i,j} + \xi_i R_{i-1,j} - \gamma_j R_{i,j-1},$$

with initial values

$$R_{i,-1} = 0, \quad R_{i,0} = L[q_i] = \nu_i, \quad i \in \mathbb{N}_0.$$

Moreover, the coefficients in the three-term recurrence relation (36) are given by

$$\beta_i = \eta_i + \frac{R_{i+1,i}}{R_{i,i}} - \frac{R_{i,i-1}}{R_{i-1,i-1}}, \tag{39}$$

and

$$\gamma_i = \frac{R_{i,i}}{R_{i-1,i-1}}. \tag{40}$$

Proof. Let A be the ULT matrix satisfying

$$\overrightarrow{q} = A \overrightarrow{p}.$$

Then,

$$R = L\left[\overrightarrow{q} \overrightarrow{p}^T\right] = L\left[A \overrightarrow{p} \overrightarrow{p}^T\right] = AH. \tag{41}$$

Hence, R is a lower triangular matrix and

$$R_{i,i} = h_i. \tag{42}$$

Using (31) and (37), we have

$$T \overrightarrow{q} \overrightarrow{p}^T = x \overrightarrow{q} \overrightarrow{p}^T = \overrightarrow{q} x \overrightarrow{p}^T = \overrightarrow{q} \overrightarrow{p}^T J^T,$$

and therefore

$$TR = L\left[T \overrightarrow{q} \overrightarrow{p}^T\right] = L\left[\overrightarrow{q} \overrightarrow{p}^T J^T\right] = R J^T.$$

Using (29) and (38), we get

$$R_{i+1,j} + \eta_i R_{i,j} + \xi_i R_{i-1,j} = R_{i,j+1} + \beta_j R_{i,j} + \gamma_j R_{i,j-1}. \tag{43}$$

Since R is a lower triangular matrix, we have

$$R_{i,j} = 0, \quad i < j, \tag{44}$$

and setting $i = j - 1$ in (43), we obtain

$$\gamma_j = \frac{R_{j,j}}{R_{j-1,j-1}}. \tag{45}$$

Note that from (42) and (45) we have

$$\gamma_j = \frac{h_j}{h_{j-1}},$$

in agreement with (30).

If we set $i = j$ in (43) and use (45) and (44), we obtain

$$\beta_j = \eta_j + \frac{R_{j+1,j} - \gamma_j R_{j,j-1}}{R_{j,j}} = \eta_j + \frac{R_{j+1,j}}{R_{j,j}} - \frac{R_{j,j-1}}{R_{j-1,j-1}}.$$

Finally, solving for $R_{i,j+1}$ in (43), we get

$$R_{i,j+1} = R_{i+1,j} + (\eta_i - \beta_j) R_{i,j} + \xi_i R_{i-1,j} - \gamma_j R_{i,j-1}.$$

■

Example 25 *Charlier polynomials. The falling factorial polynomials satisfy the 3-term recurrence relation (3). Comparing with (38), we see that*

$$\eta_n = n, \quad \xi_n = 0,$$

and therefore

$$T_{i,j} = \delta_{i+1,j} + i\delta_{i,j}.$$

Using (41), we get

$$R_{i,j} = \sum_{k=0}^{\infty} C_{i,k} H_{k,j} = C_{i,j} h_j = \binom{i}{j} z^{i-j} j!\ z^j e^z = e^z j! \binom{i}{j} z^i.$$

Finally, using (39) and (40) we obtain [1]

$$\beta_n = n + \frac{R_{n+1,n}}{R_{n,n}} - \frac{R_{n,n-1}}{R_{n-1,n-1}} = n + z, \tag{46}$$

and

$$\gamma_n = \frac{R_{n,n}}{R_{n-1,n-1}} = nz. \tag{47}$$

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