



The Sage package `comb_walks` for Walks in the Quarter Plane

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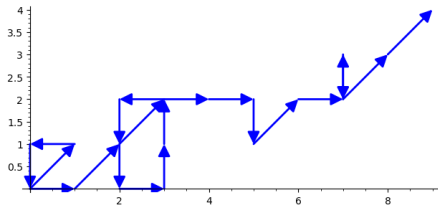
Outline

- 1 Walks in the Quarter Plane
- 2 How to use `comb_walks`
- 3 What to do with `comb_walks`
- 4 Conclusions



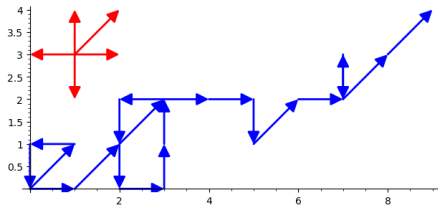
Walks in the Quarter Plane

Walks in the Quarter Plane



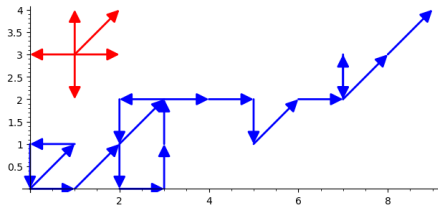
- Never cross the X and Y axis.

Walks in the Quarter Plane



- Never cross the X and Y axis.
- Steps taken from a fixed set of valid steps.

Walks in the Quarter Plane



- Never cross the X and Y axis.
- Steps taken from a fixed set of valid steps.
- Length of the walk = Number of steps

Combinatorial problem

A counting problem

- Given a set of valid steps \mathcal{S} , how many walks are there that start from $(0,0)$ and end at (i,j) after n steps taken from \mathcal{S} ?
- We call that number $q_{i,j,n}$.

Combinatorial problem

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$$Q(x, y, t) = \sum_{i,j,n \in \mathbb{N}} q_{i,j,n} x^i y^j t^n.$$



A classification problem

Given a set \mathcal{S} , what algebraic or differential properties does $Q(x, y, t)$ have?

A classification problem

Given a set \mathcal{S} , what algebraic or differential properties does $Q(x, y, t)$ have?

- **Rational:** there are $N(x, y, t), D(x, y, t) \in \mathbb{Q}[x, y, t]$ such that

$$Q(x, y, t) = \frac{N(x, y, t)}{D(x, y, t)}.$$

- **Algebraic:** there is $P(Z) \in \mathbb{Q}(x, y, t)[Z]$ such that $P(Q(x, y, t)) = 0$.
- **D-finite:** $Q(x, y, t)$ satisfies linear differential equations w.r.t. ∂_x, ∂_y and ∂_t with coefficients in $\mathbb{Q}[x, y, t]$.
- **D-algebraic:** $Q(x, y, t)$ satisfies non-linear differential equations w.r.t. ∂_x, ∂_y or ∂_t with coefficients in $\mathbb{Q}[x, y, t]$.
- **D-transcendental:** any other case.



The kernel equation

Kernel polynomial

Fixed set \mathcal{S} of valid steps:

$$K(x, y, t) = xy \left(1 - t \sum_{(a,b) \in \mathcal{S}} x^a y^b \right).$$

The kernel equation

Kernel polynomial

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Functional equation

$$\begin{aligned} Q(x, y, t)K(x, y, t) &= xy - Q(0, 0, t)K(0, 0, t) \\ &\quad + Q(x, 0, t)K(x, 0, t) + Q(0, y, t)K(0, y, t). \end{aligned}$$



Tools used for classification

- Sections of $Q(x, y, t)K(x, y, t)$: study

$$F_1(x, t) = Q(x, 0, t)K(x, 0, t), \quad F_2(y, t) = Q(0, y, t)K(0, y, t).$$

- Algebraic geometry: study the curve in $\mathbb{P} \times \mathbb{P}$:

$$E_t = \left\{ (x_0 : x_1, y_0 : y_1) : K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t\right) = 0 \right\}.$$

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- 1 Case of a curve of genus 0: rational parametrization.
- 2 Case of a curve of genus 1: elliptic curve.



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- 1 Case of a curve of genus 0: rational parametrization.
- 2 **Case of a curve of genus 1: elliptic curve.**



How to get and use `comb_walks`



Git repository

Code hosted in GitLab Inria

https://gitlab.inria.fr/discretewalks/comb_walks

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https://gitlab.inria.fr/discretewalks/comb_walks

Installation via *pip*

Recommended to install all the dependencies:

```
sage -pip [--user] install git+https://gitlab.inria.fr/discretewalks/comb_walks.git
```



Documentation and Demo

Documentation available online

https://discretewalks.gitlabpages.inria.fr/comb_walks/docs/

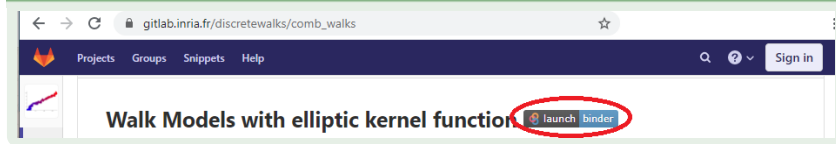


Documentation and Demo

Documentation available online

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Easy accessible demo using Binder



Documentation and Demo

jupyter comb_walks_demo (unsaved changes)

File Edit View Insert Cell Kernel Widgets Help

Not Trusted SageMath 9.0

Walks in the Quarter Plane in Sage

The package comb_walks

1. Introduction to walks in the plane

A walk in the lattice \mathbb{Z}^2 is a sequence of points (P_0, \dots, P_n) . We say that P_0 is the origin of the walk and P_n is its ending point, that n is its length and that all the differences $s_i = P_i - P_{i-1} \in \mathbb{Z}^2$ for $1 \leq n$ are its steps.

Given a set $S \subset \mathbb{Z}^2$ whose elements we call *valid steps*, we can consider walks in the lattice plane that only have steps in such subset, i.e., walks where $s_i \in S$ for all $1 \leq i \leq n$. We call the resulting set of walks a *model of walks*.

The following combinatorial problem was considered:

- **Walks on the plane:** given a set $S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ (which are called *small steps*), how many walks are there from the origin $(0, 0)$ to the point (i, j) using n steps taken from S ?

To study this problem, we collect the numbers $g_{i,j,n}$ in a generating series:

$$G(x, y, t) = \sum_{i,j \in \mathbb{Z}} \sum_{n \in \mathbb{N}} g_{i,j,n} x^i y^j t^n.$$

The classification of different models according to the algebraic and differential nature of the function $G(x, y, t)$ is done by using the recurrence relation

$$g_{i,j,n} = \sum_{(a,b) \in S} g_{i-a,j-b,(n-1)},$$

which holds for all different models. Moreover, for each $(a, b) \in \mathbb{Z}^2$ we have

$$\begin{aligned} t^a x^b G(x, y, t) &= \sum_{i,j \in \mathbb{Z}} \sum_{n \in \mathbb{N}} g_{i,j,n} x^{i+a} y^{j+b} t^{n+1} \\ &= \sum_{i,j \in \mathbb{Z}} \sum_{n \geq 1} g_{(i-a),(j-b),(n+1)} x^i y^j t^n. \end{aligned}$$

Now, if we sum the left-hand sides of the previous equation over all $(a, b) \in S$, we can simplify the right-hand side using the recurrence equation. This leads to the following equation:

$$G(x, y, t) = \frac{1}{1 - tS(x, y)},$$

where $S(x, y)$ is the so-called *step polynomial* of the model S and it is defined as follows:



What can we do with `comb_walks`



Main user data structure

WalkModel

- Each instance of the class represents a model of walks.
- The constructor inputs the set of valid steps.

```
In [2]: WalkModel((1,0),(0,1),(-1,-1))
```

```
Out[2]: Walk Model with steps: ((1, 0), 1), ((0, 1), 1), ((-1, -1), 1)
```



Main user data structure

WalkModel

- Each instance of the class represents a model of walks.
- The constructor inputs the set of valid steps.
- Possible to use letters for the steps: N, S, E, W, etc.

```
In [3]: WalkModel(N,E,SW)
```

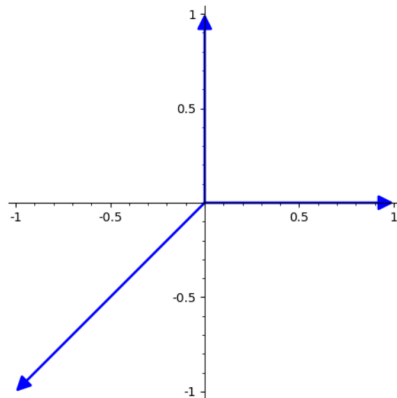
```
Out[3]: Walk Model with steps: ((0, 1), 1), ((1, 0), 1), ((-1, -1), 1)
```



Plotting methods

```
In [6]: plot(WalkModel(N,E,SW))
```

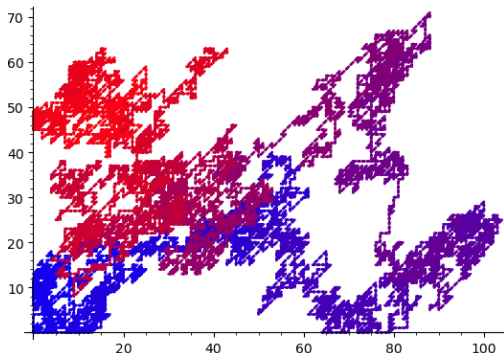
Out[6]:



Plotting methods

```
In [12]: # A long example of the model  
m.plot_random_walk(10000, arrowsize=0.001)
```

Out[12]:



Methods for geometric objects

Kernel function

The function K in \mathbb{P}^2 :

```
In [13]: WalkModel(N,NE,W,SW).kernel()
```

```
Out[13]:  $(-t)x^2y^2 + (-t)xy^2z + xyz^2 + (-t)yz^3 + (-t)z^4$ 
```



Methods for geometric objects

Kernel function

The function K in $\mathbb{P} \times \mathbb{P}$:

```
In [14]: WalkModel(N,NE,W,SW).kernel('P')
```

```
Out[14]:  $(-t)x_0^2y_0^2 + (-t)x_0x_1y_0^2 + x_0x_1y_0y_1 + (-t)x_1^2y_0y_1 + (-t)x_1^2y_1^2$ 
```



Methods for geometric objects

Kernel function

The function K in $\mathbb{P} \times \mathbb{P}$:

The automorphism τ

Map that changes first the y coordinate and then the x coordinate:

```
In [20]: WalkModel(N,NE,W,SW).tau('P')
```

```
Out[20]: Scheme endomorphism of Closed subscheme of Product of projective spaces P^1 x P^1 over Fraction Field of Univariate Polynomial
Ring in t over Rational Field defined by:
(-t)*x0^2*y0^2 + (-t)*x0*x1*y0^2 + x0*x1*y0*y1 + (-t)*x1^2*y0*y1 + (-t)*x1^2*y1^2
Defn: Defined by sending (x0 : x1 , y0 : y1) to
(-x0^2*y0 - x0*x1*y0 + t*x0*x1*y1 + t*x1^2*y1 : (-t)*x1^2*y1 , -x1^2*y1 : -x0^2*y0 - x0*x1*y0).
```



Methods for geometric objects

The function $b_i := \tau(F_i) - F_i$

```
In [23]: WalkModel(N,NE,W,SW).b(1)
```

```
Out[23]: 
$$\frac{\left(\frac{-1}{t}\right)x^2y + \left(\frac{-1}{t}\right)xy + 2x + 1}{-x^2y - xy}$$

```

```
In [24]: WalkModel(N,NE,W,SW).b(2)
```

```
Out[24]: 
$$\frac{x^2y^2 + xy^2 - 1}{-xy - y}$$

```



Telescoping functions over the curve

Poles of rational functions on the kernel curve

```
In [28]: model = WalkModel(NE,S,W)
         model.poles(model.b(2))
```

```
Out[28]: [(0 : 1 , 1 : 0), (1 : 0 , 0 : 1), (0 : 1 , 0 : 1)]
```



Telescoping functions over the curve

Poles of rational functions on the kernel curve

```
In [28]: model = WalkModel(NE,S,W)
         model.poles(model.b(2))
```

```
Out[28]: [(0 : 1 , 1 : 0), (1 : 0 , 0 : 1), (0 : 1 , 0 : 1)]
```

Telescoping functions over the curve

Compute L and g such that $L \cdot b = \tau(g) - g$:

```
In [31]: L,g = model.telescoping(model.b(2)(x=x0/x1,y=y0/y1))
         print(L) # It is (0,1)
         tau = pullback(model.tau('P'))
         # Checking the equation on the curve
         simplify_rational_variety(model.derivative(model.b(2))(x=x0/x1,y=y0/y1) - (tau(g)-g), model.curve('P')) == 0

0 1
```

```
Out[31]: True
```



Telescoping functions over the curve

Poles of rational functions on the kernel curve

```
In [28]: model = WalkModel(NE,S,W)
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```

```
Out[28]: [(0 : 1 , 1 : 0), (1 : 0 , 0 : 1), (0 : 1 , 0 : 1)]
```

Telescoping functions over the curve

Compute L and g such that $L \cdot b = \tau(g) - g$. In this case:

$$\delta b_2 = (\tau - 1) \left(t \frac{x_0^2 y_0 + x_1^2 y_1}{x_0 x_1 y_0} \right).$$



Built-in models

- `FiniteGroup`: all models where $\text{ord}(\tau) < \infty$.
- `EllipticC`: models with infinite group and elliptic kernel.
- `NonEllipticC`: other models.



Built-in models

- `FiniteGroup`: all models where $\text{ord}(\tau) < \infty$.
- `EllipticC`: models with infinite group and elliptic kernel.
- `NonEllipticC`: other models.

Exhaustive website

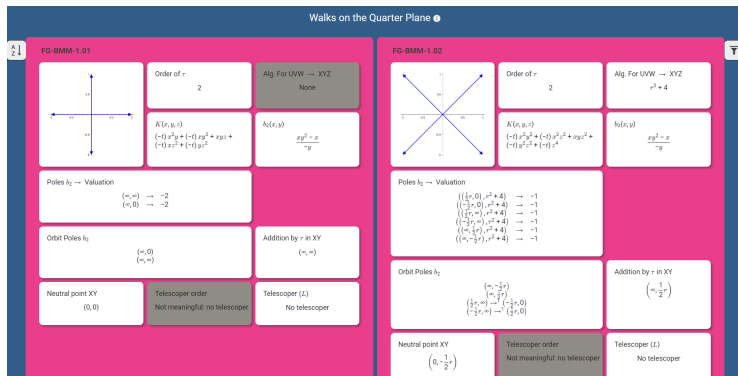
The functionality of the package has been applied to all those models. The results can be found in the webpage

https://discretewalks.gitlabpages.inria.fr/comb_walks/



The website

https://discretewalks.gitlabpages.inria.fr/comb_walks/




The website

https://discretewalks.gitlabpages.inria.fr/comb_walks/

Walks on the Quarter Plane ●

FG-BMM-1.01



Order of r

2

Alg. For UVW \rightarrow XYZ

None

$K(x, y, z)$

$$\{(-1)x^2y + (-1)xy^2 + xy\} + \{(-1)x^2z + (-1)yz^2 + xz\}$$

$h_2(x, y)$

$$\frac{xy^2 - x}{-y}$$

Poles $h_2 \rightarrow$ Valuation

$$\begin{aligned} (w, w) &\rightarrow -2 \\ (w, 0) &\rightarrow -2 \end{aligned}$$

Orbit Poles h_2

$$(w, 0)$$

Neutral point XY

$$(0, 0)$$


Telescoper order

Not meaningful: no telescoper

Telescoper (L)

No telescoper

FG-BMM-1.02



Order of r

2

Alg. For UVW \rightarrow XYZ

None

$K(x, y, z)$

$$\{(-1)x^2y^2 + (-1)xy^2z\} + \{(-1)xy^2z^2 + (-1)xyz^2\}$$

Poles $h_2 \rightarrow$ Valuation

$$\begin{aligned} ((\frac{1}{2}r, 0), r^2 + 4) &\rightarrow -1 \\ ((-\frac{1}{2}r, 0), r^2 + 4) &\rightarrow -1 \\ ((\frac{1}{2}r, w), r^2 + 4) &\rightarrow -1 \\ ((-\frac{1}{2}r, w), r^2 + 4) &\rightarrow -1 \\ ((w, \frac{1}{2}r), r^2 + 4) &\rightarrow -1 \\ ((w, -\frac{1}{2}r), r^2 + 4) &\rightarrow -1 \end{aligned}$$

Orbit Poles h_2

$$\begin{pmatrix} w, -\frac{1}{2}r \\ w, \frac{1}{2}r \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{1}{2}r, 0 \\ \frac{1}{2}r, 0 \end{pmatrix}$$

Neutral point XY

$$(0, -\frac{1}{2}r)$$

Telescoper order

Not meaningful: no telescoper

Telescoper (L)

No telescoper

22 / 78

Search

Filter by type of model

- ☒ Finite group \rightarrow For UVW \rightarrow XYZ
- ☐ Non-elliptic curve
- ☐ Elliptic curve

Show attributes:


- ☒ Order of r
- ☒ Alg. For UVW \rightarrow XYZ
- ☒ $K(x, y, z)$
- ☐ $K(x, w)$
- ☐ $K(x, y, w)$
- ☒ $h_2(x, y)$
- ☒ Poles $h_2 \rightarrow$ Valuation
- ☐ Orbit Poles h_2
- ☐ Orbit sum h_2
- ☐ Poles orbit sum
- ☒ Addition by r in XY
- ☐ Addition by r in UV
- ☒ Neutral point XY
- ☐ xy
- ☐ yz
- ☐ Telescoper order
- ☒ Telescoper (L)
- ☐ Numerator of Certificate (g)
- ☐ Denominator of Certificate (g)

The website

https://discretewalks.gitlabpages.inria.fr/comb_walks/

Walks on the Quarter Plane ●

FG-BMM-1.01



Order of τ

2

Alg. For UVW \rightarrow XYZ

None

$K(x, y, z)$

$$(-t)x^2y + (-t)xy^2 + xy; +$$

$$(-t)xz^2 + (-t)yz^2 +$$

$h_2(x, y)$

$$xy^2 - x$$

$$-y$$

Poles $h_2 \rightarrow$ Valuation

$$(\omega, \omega) \rightarrow -2$$

$$(\omega, 0) \rightarrow -2$$

Orbit Poles h_2

$$(\omega, 0)$$

$$(\omega, \omega)$$

Addition by τ in XY

$$(\omega, \omega)$$

Neutral point XY

$$(0, 0)$$


Telescoper order

Not meaningful: no telescoper

Telescoper (L)

No telescoper

FG-BMM-1.02



Order of τ

2

Alg. For UVW \rightarrow XYZ

None

$K(x, y, z)$

$$(-t)x^2y^2 + (-t)xy^2z$$

$$(-t)y^2z^2 + (-t)z^4$$

$h_2(x, y)$

$$xy^2 - x$$

$$-y$$

Poles $h_2 \rightarrow$ Valuation

$$\left(\left(\frac{1}{2}r, 0\right), r^2 + 4\right) \rightarrow -1$$

$$\left(\left(-\frac{1}{2}r, 0\right), r^2 + 4\right) \rightarrow -1$$

$$\left(\left(\frac{1}{2}r, \omega\right), r^2 + 4\right) \rightarrow -1$$

$$\left(\left(-\frac{1}{2}r, \omega\right), r^2 + 4\right) \rightarrow -1$$

$$\left(\left(\omega, \frac{1}{2}r\right), r^2 + 4\right) \rightarrow -1$$

$$\left(\left(\omega, -\frac{1}{2}r\right), r^2 + 4\right) \rightarrow -1$$

Orbit Poles h_2

$$\left(\frac{1}{2}r, \omega\right) \rightarrow -1$$

$$\left(-\frac{1}{2}r, \omega\right) \rightarrow -1$$

$$\left(\omega, \frac{1}{2}r\right) \rightarrow -1$$

$$\left(\omega, -\frac{1}{2}r\right) \rightarrow -1$$

Neutral point XY

$$\left(0, -\frac{1}{2}r\right)$$

Telescoper order

Not meaningful: no telescoper

Telescoper (L)

No telescoper

Search

22 / 78

Filter by type of model

☒ Finite group \rightarrow For UVW \rightarrow XYZ

☐ Non-elliptic curve

☐ Elliptic curve

Show attributes:

☒ Order of τ

☒ Alg. For UVW \rightarrow XYZ

☒ $K(x, y, z)$

☐ $K(x_0, y_0)$

☐ $K(x_0, y_0, z_0)$

☒ $h_2(x, y)$

☒ Poles $h_2 \rightarrow$ Valuation

☒ Orbit Poles h_2

☐ Orbit sum h_2

☐ Poles orbit sum

☒ Addition by τ in XY

☐ Addition by τ in UV

☒ Neutral point XY

☐ h_2

☐ h_2

☒ Telescoper order

☒ Telescoper (L)

☐ Numerator of Certificate (g)

☐ Denominator of Certificate (g)



The website

https://discretewalks.gitlabpages.inria.fr/comb_walks/

Walks on the Quarter Plane

FG-BMM-1.01

Order of τ : 2

Alg. For UVW \rightarrow XYZ: None

$K(x, y, z)$
 $(-t)x^2y + (-t)xy^2 + xyz + (-t)yz^2 + (-t)y^2z$

$b_2(x, y)$
 $\frac{xy^2 - x}{-y}$

Orbit Poles b_2
 $(\infty, 0)$
 (∞, ∞)

Addition by τ in XY: (∞, ∞)

Neutral point XY: $(0, 0)$

Telescopier order: Not meaningful: no telescopier

Telescopier (L): No telescopier

FG-BMM-1.02

Order of τ : 2

$K(x, y, z)$
 $(-t)x^2y^2 + (-t)x^2z^2 + (-t)y^2z^2 + (-t)z^4$

Orbit Poles b_2
 $(\infty, -\frac{1}{2}\tau)$
 $(\infty, \frac{1}{2}\tau)$
 $(\frac{1}{2}\tau, \infty) \rightarrow (-\frac{1}{2}\tau, 0)$
 $(-\frac{1}{2}\tau, \infty) \rightarrow (-\frac{1}{2}\tau, 0)$

Neutral point XY: $(0, -\frac{1}{2}\tau)$

Telescopier order: Not meaningful: no telescopier

FG-BMM-1.03

Order of τ : 2

Alg. For UVW \rightarrow XYZ: $\tau^2 + 4$

FG-BMM-1.04

Order of τ : 2

Alg. For UVW \rightarrow XYZ: $\tau^2 + 3$

Search: 22 / 78

Filter by type of model:
☒ Finite group
☐ Non-elliptic curve
☐ Elliptic curve

Show attributes:
☒ Order of τ
☐ $\tau(x_0, y_0, z_0, w_0)$
☒ Alg. For UVW \rightarrow XYZ
☒ $K(x, y, z)$
☐ $K(x_0, y_0, z_0)$
☐ $K(x_0, y_0, w_0)$
☐ $K(x_0, y_0, w_0, w_1)$
☒ $b_2(x, y)$
☐ Poles $b_2 \rightarrow$ Valuation
☒ Orbit Poles b_2
☐ Orbit sum b_2
☐ Poles orbit sum
☒ Addition by τ in XY
☒ Addition by τ in UV
☒ Neutral point XY
☐ τ^2
☐ τ^3
☒ Telescopier order
☒ Telescopier (L)
☐ Numerator of Certificate (μ)
☐ Denominator of Certificate (μ)



The website

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Walks on the Quarter Plane ●

NE-DHRS-5

Order of r
 $+\infty$

$K(x, y, z)$
 $(-t)x^2y + (-t)xy^2 + (-t)x^2z + xyz + (-t)y^2z$

Neutral point XY
Not meaningful: not elliptic

$h_2(x, y)$
 $\frac{x^2y^2 - x^3 + xy^2}{-xy - y}$

Telescoper order

Telescoper (L)

wIA.05

Order of r
 $+\infty$

$h_2(x, y)$
 $\frac{x^3y^2 + x^2y^2 - x}{-x^2y - xy}$

Telescoper order
Not meaningful: no telescoper

Telescoper (L)

No telescoper

wIB.5

Order of r
 $+\infty$

$K(x, y, z)$
 $(-t)x^2y^2 + xyz^2 + (-t)y^2z^2 + (-t)xz^2 + (-t)yz^2 + (-t)z^3$

Neutral point XY
 $(-1, 0)$

$h_2(x, y)$
 $\frac{x^3y^2 + xy^2 - x^2 - x}{-x^2y - y}$

wIA.5

Order of r
 $+\infty$

$h_2(x, y)$
 $\frac{x^2y^2 + xy^2 - x^2 - y - 1}{-xy - y}$

Neutral point XY
 $(0, \infty)$

Filter by type of model:

- ☐ Finite group
- ☒ Non-elliptic curve
- ☐ Elliptic curve

Show attributes:

- ☒ Order of r
 - ☐ $r(x_0, y_1, y_0, y_1)$
 - ☐ Alg. For UVW \rightarrow XYZ
 - ☒ $K(x, y, z)$
 - ☐ $K(x, y, w)$
 - ☐ $K(x_0, x_1, y_0, y_1)$
 - ☒ $h_2(x, y)$
 - ☐ Poles $h_2 \rightarrow$ Valuation
 - ☐ Orbit Poles h_2
 - ☐ Orbit sum h_2
 - ☐ Poles orbit sum
 - ☐ Addition by r in XY
 - ☐ Addition by w in UV
 - ☒ Neutral point XY
 - ☐ y_0
 - ☐ y_1
 - ☒ Telescoper order
 - ☒ Telescoper (L)
 - ☐ Numerator of Certificate (y_0)
 - ☐ Denominator of Certificate (y_0)



Conclusions

Conclusions

Achievements

- A framework to classify models of walks in the quarter plane
- Telescoping method over elliptic kernel functions
- Visualization of the results over classical examples

Future work

- Develop a constructive method for obtaining algebraic and differential equations
- Study the performance and complexity of the current code
- Adapt the current implementation to wider use



Thank you!

Results and documentation:

- https://discretewalks.gitlabpages.inria.fr/comb_walks/
- https://discretewalks.gitlabpages.inria.fr/comb_walks/docs/

GitLab repository:

- https://gitlab.inria.fr/discretewalks/comb_walks

