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# The Sage package comb\_walks

for Walks in the Quarter Plane

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Der Wissenschaftsfonds.



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# Outline

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# Walks in the Quarter Plane

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# Walks in the Quarter Plane



• Never cross the X and Y axis.

Walks in the Quarter Plane  $_{\circ \circ \circ \circ \circ \circ}$ 

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# Walks in the Quarter Plane



- Never cross the X and Y axis.
- Steps taken from a fixed set of valid steps.

Walks in the Quarter Plane  $_{\circ \circ \circ \circ \circ \circ}$ 

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#### Walks in the Quarter Plane



- Never cross the X and Y axis.
- Steps taken from a fixed set of valid steps.
- Length of the walk = Number of steps

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## Combinatorial problem

#### A counting problem

- Given a set of valid steps S, how many walks are there that start from (0,0) and end at (i, j) after n steps taken from S?
- We call that number  $q_{i,j,n}$ .

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## Combinatorial problem

#### A counting problem

- Given a set of valid steps S, how many walks are there that start from (0,0) and end at (i, j) after n steps taken from S?
- We call that number  $q_{i,j,n}$ .

$$Q(x,y,t)=\sum_{i,j,n\in\mathbb{N}}q_{i,j,n}x^iy^jt^n.$$

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## A classification problem

# Given a set $\mathcal{S}$ , what algebraic or differential properties does Q(x, y, t) have?



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# A classification problem

Given a set  $\mathcal{S}$ , what algebraic or differential properties does Q(x, y, t) have?

• Rational: there are  $N(x, y, t), D(x, y, t) \in \mathbb{Q}[x, y, t]$  such that

$$Q(x,y,t)=\frac{N(x,y,t)}{D(x,y,t)}.$$

- Algebraic: there is  $P(Z) \in \mathbb{Q}(x, y, t)[Z]$  such that P(Q(x, y, t)) = 0.
- **D-finite**: Q(x, y, t) satisfies linear differential equations w.r.t.  $\partial_x$ ,  $\partial_y$  and  $\partial_t$  with coefficients in  $\mathbb{Q}[x, y, t]$ .
- D-algebraic: Q(x, y, t) satisfies non-linear differential equations w.r.t. ∂<sub>x</sub>, ∂<sub>y</sub> or ∂<sub>t</sub> with coefficients in Q[x, y, t].
- D-transcendental: any other case.

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## The kernel equation

#### Kernel polynomial

Fixed set  $\mathcal{S}$  of valid steps:

$$\mathcal{K}(x,y,t) = xy\left(1-t\sum_{(a,b)\in\mathcal{S}}x^ay^b\right).$$

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## The kernel equation

#### Kernel polynomial

Fixed set  $\mathcal{S}$  of valid steps:

$$\mathcal{K}(x,y,t) = xy\left(1 - t\sum_{(a,b)\in\mathcal{S}} x^a y^b\right).$$

#### Functional equation

$$\begin{aligned} Q(x,y,t) & \mathcal{K}(x,y,t) = xy - Q(0,0,t) \mathcal{K}(0,0,t) \\ & + Q(x,0,t) \mathcal{K}(x,0,t) + Q(0,y,t) \mathcal{K}(0,y,t). \end{aligned}$$

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## Tools used for classification

• Sections of Q(x, y, t)K(x, y, t): study

$$F_1(x,t) = Q(x,0,t)K(x,0,t), \ F_2(y,t) = Q(0,y,t)K(0,y,t).$$

• Algebraic geometry: study the curve in  $\mathbb{P}\times\mathbb{P}$ :

$$E_t = \left\{ (x_0 : x_1, y_0 : y_1) : K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t\right) = 0 \right\}.$$

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## Tools used for classification

• Sections of Q(x, y, t)K(x, y, t): study

$$F_1(x,t) = Q(x,0,t)K(x,0,t), \ F_2(y,t) = Q(0,y,t)K(0,y,t).$$

• Algebraic geometry: study the curve in  $\mathbb{P}\times\mathbb{P}$ :

$$E_t = \left\{ (x_0 : x_1, y_0 : y_1) : K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t\right) = 0 \right\}$$

Case of a curve of genus 0: rational parametrization.Case of a curve of genus 1: elliptic curve.

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#### Tools used for classification

• Sections of Q(x, y, t)K(x, y, t): study

$$F_1(x,t) = Q(x,0,t)K(x,0,t), \ F_2(y,t) = Q(0,y,t)K(0,y,t).$$

• Algebraic geometry: study the curve in  $\mathbb{P}\times\mathbb{P}$ :

$$E_t = \left\{ (x_0 : x_1, y_0 : y_1) : K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t\right) = 0 \right\}$$

Case of a curve of genus 0: rational parametrization.
Case of a curve of genus 1: elliptic curve.

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# How to get and use comb\_walks

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# Git repository

#### Code hosted in GitLab Inria

#### https://gitlab.inria.fr/discretewalks/comb\_walks



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# Git repository

## Code hosted in GitLab Inria

https://gitlab.inria.fr/discretewalks/comb\_walks

Installation via pip

Recommended to install all the dependencies:

sage -pip [--user] install git+https://gitlab.inria.fr/discretewalks/comb\_walks.git

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#### Documentation and Demo

#### Documentation available online

https://discretewalks.gitlabpages.inria.fr/comb\_walks/docs/



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#### Documentation and Demo

#### Documentation available online

https://discretewalks.gitlabpages.inria.fr/comb\_walks/docs/

Easy accessible demo using Binder					
$\leftarrow$ $\rightarrow$ C a gitlab.inria.fr/discretewalks/comb_walks	:				
🔶 Projects Groups Snippets Help	Q 🕜 ~ Sign in				
Walk Models with elliptic kernel function					

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# Documentation and Demo

<b>Walks in the Quarter Plane in Sage</b> <b>The package comb_walks</b> <b>Introduction to walks in the plane</b> Awake the lather $Z^2$ is a sequence of points $(P_{1,},P_{r})$ we say that $P_0$ is the origin of the walk and $P_n$ is the ending point, that <i>n</i> is its length and the former say $= P_1 - P_{1-1} \in Z^2$ for $1 \le n = n$ is stages. Given a set $S \in Z^2$ whose been tests we call weld attack, we can consider walks in the lather plane that only have steps in such subset, i.e., walks where $v_0 \in S$ for all $\le i \le n$ . We can be returning set of walks and of or walks. The transmitted in the plane in the same set of $S \in [-1, 0, 1]^2 \setminus \{(0, 0)\}$ (which are called annal steps), how many walks are there from the origin $(0, 0)$ to the $i_1$ ( <i>i</i> ) using a steps latent from $S^2$ . The classification of different models according to the algebraic and differential nature of the function $Q(x, y, t)$ is done by using the recurrence relation $S_{i_1} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_2} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_2} = S_{i_1} = S_{i_1} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_1} = S_{i_2} = S_{i_1} = S_{i_1} = S_{i_1} = S_{i_2} = S_{i_1} = S_{i_1$	File Edit	View Insert Cell Kernel Widgets Help	Not Truste	d SapeMath 9
Walks in the Quarter Plane in Sage The package comb_walks Introduction to walks in the plane A wake the black $Z^2$ is a sequence of points $(P_{n-1}, P_{n})$ with say that $P_0$ is the origin of the wake and $P_n$ is the ending point, that <i>n</i> is its length and the differences $z_1 = P_n - P_n + Z^2$ is of 1 $z$ are in stars. The blacking combinating pole mass consider walks in the lattice plane that only have steps in such subset, i.e., wake where $z_1 \in S$ for all $z \neq 1$ solve a dimension of a stars. The blacking combinating pole mass consider walks in the lattice plane that only have steps in such subset, i.e., wake where $z_1 \in S$ for all $z \neq 1$ solve a dimension of a stars. The blacking combinating pole mass consider walks are due of walks. The blacking combinating pole mass consider the black is not plane; gives a set $S \leq (-1, 0, 1)^2 \setminus (0, 0)$ (which are called small ateps), how many walks are there from the origin $(0, 0)$ to the $z_1$ $Z$ but you for problem, we called the numbers $g_{1,n}$ in a generating setset: $U = \sum_{n=1}^{n} $				- Ougement
<b>The package comb_walks</b> <b>1. Introduction to walks in the plane</b> Wash in the lattice $2^2$ is a sequence of points $(P_0,, P_n)$ . We say that $P_0$ is the origin of the walk and $P_n$ is the ending point, that $n$ is its length and the differences $p_1 = P_{-1} = 2^2$ . Even $(1 \le n \le $	a + o -			
The package comb_walks 1. Introduction to walks in the plane Mark in the lattice $Z^2$ is a sequence of points $(P_1, \dots, P_n)$ . We say that $P_0$ is the origin of the walk and $P_n$ is the ending point, that $n$ is its length and the differences $n \in P_1 - P_{-1} \in Z^2$ (or $1 \le n$ are its steps: Given a task $S \ge Z^2$ indices indernet were call added walks in the lattice plane that only have steps in such subset, i.e., walks where $n \le S$ for all $\le 1 \le n$ . We call the resulting set of walks a model of walks. The tolowing combinatorial problem was considered with some the problem, we collect the numbers $g_{1/n}$ in a generating user: $(L_1, L_2) = \sum_{k \le N} E_k (L_k, L_k) = \sum_{k \ge N} E_k (L_k, L_k)$ , is done by using the recurrence relation $E_k (L_k) = \sum_{k \ge N} E_k (L_k) e^{-L_k} (L_k)$ . The classification of different models according to the algebra and after the function $G(x_k, y, t_k)$ is done by using the recurrence relation $E_k (L_k) = \sum_{k \ge N} E_k (L_k) e^{-L_k} (L_k) = $				
The package comb_walks 1. Introduction to walks in the plane Mark in the lattice $Z^2$ is a sequence of points $(P_1,, P_n)$ . We say that $P_0$ is the origin of the walk and $P_n$ is the ending point, that <i>n</i> is the length and the the differences $p_1 = P_1 - P_1 = Q^2$ of $1 \le n$ are its steps: Grave and $S < Z^2$ allowed sements we call allowed sides we can consider walks in the lattice plane that only have steps in such subset, i.e., walks where $i_0 \le S$ for all $1 \le i_0 \le N$ we call the resulting set of walks a model of walks. The following combinativity problems was considered: 1. (a) using a steps taken from S? To study this problem, we callect the numbers $g_{i_0 k}$ in a generating user: $C(x, y, \alpha) = \sum_{\substack{k \in I \\ k \in I \le N}} K_{i_0} + M_{i_0} + M_{i_0$				
The package comb_walks 1. Introduction to walks in the plane Awak in the lattice $Z^1$ is a sequence of points $(P_0,, P_n)$ . We say that $P_0$ is the origin of the walk and $P_a$ is the ending point, that $n$ is its length and the the differences $a_1 = P_1 - P_{-1} \in Z^2$ for $1 \le a$ are its steps: Given a cast $S \in Z^3$ mixed sements set call all objects, we can consider walks in the lattice plane that only have steps in such subset, i.e., walks where $a_2 \le S$ for all $\le 1 \le a$ . We call the resulting set of walks a model of walks. The bilowing combinativity problem was considered 1. $(i,j)$ using a subgest taken from $S^2$ . To study this problem, we collect the numbers $g_{ijk}$ are generating series: $G(x_1, x_1) = \sum_{i \in J \\ i \in S \ i$				
1. Introduction to walks in the plane A walk in the lattice $Z^2$ is a sequence of points $(P_0,, P_d)$ . We say that $P_0$ is the origin of the walk and $P_n$ is the ending point, that $n$ is its length and the differences $a_1 = P_1 - P_{-1} \in Z^2$ for $1 \le n$ are to stags. Given the set of $S \subseteq Z^2$ have elements use call used for say, use on consider walks in the lattice plane that only have steps in such subset, i.e., walks where $a_i \in S$ for all $1 \le i \le n$ . We call the resulting set of walks a model of walks. The following combinatorial problem was considered: 1. Whise an the galaxies elements use call used for a generating series: The classification of different models according to the algebraic and differential nature of the function $G(x, y, \alpha)$ is done by using the recurrence relation: $R_{M_1} = \sum_{k \in K} R_{M_1} a_k^{k-1} e^{k-1}$ . The classification of different models according to the algebraic and differential nature of the function $G(x, y, \alpha)$ is done by using the recurrence relation: $R_{M_2} = \sum_{k \in K} R_{M_2} a_k^{k-1} e^{k-1}$ . The classification of different models according to the algebraic and differential nature of the function $G(x, y, \alpha)$ is done by using the recurrence relation: $R_{M_2} = \sum_{k \in K} R_{M_2} a_k^{k-1} e^{k-1} e^{k-1}$ . Nucleic for all different models. Information: $R_{M_2} = \sum_{k \in K} R_{M_2} a_{M_2} a_{M$		Walks in the Quarter Plane in Sage		
1. Introduction to walks in the plane A walk in the lattice $Z^2$ is a sequence of points $(P_0,, P_d)$ . We say that $P_0$ is the origin of the walk and $P_n$ is the ending point, that $n$ is its length and the differences $a_1 = P_1 - P_{-1} \in Z^2$ for $1 \le n$ are to stags. Given the set of $S \subseteq Z^2$ have elements use call used for say, use on consider walks in the lattice plane that only have steps in such subset, i.e., walks where $a_i \in S$ for all $1 \le i \le n$ . We call the resulting set of walks a model of walks. The following combinatorial problem was considered: 1. Whise an the galaxies elements use call used for a generating series: The classification of different models according to the algebraic and differential nature of the function $G(x, y, \alpha)$ is done by using the recurrence relation: $R_{M_1} = \sum_{k \in K} R_{M_1} a_k^{k-1} e^{k-1}$ . The classification of different models according to the algebraic and differential nature of the function $G(x, y, \alpha)$ is done by using the recurrence relation: $R_{M_2} = \sum_{k \in K} R_{M_2} a_k^{k-1} e^{k-1}$ . The classification of different models according to the algebraic and differential nature of the function $G(x, y, \alpha)$ is done by using the recurrence relation: $R_{M_2} = \sum_{k \in K} R_{M_2} a_k^{k-1} e^{k-1} e^{k-1}$ . Nucleic for all different models. Information: $R_{M_2} = \sum_{k \in K} R_{M_2} a_{M_2} a_{M$		The package comb walks		
A walk in the lattice $Z^2$ is a sequence of points $\{P_0,, P_d\}$ . We say that $P_0$ is the origin of the walk and $P_a$ is the ending of the walk are there from the origin (0, 0) to the $y_i \in S$ for all $1 \le i \le n$ . We call the resulting set of walks a model of walks. The following combinatorial problem was considered: <b>1</b> Notices on the places: given as at $S < (-1, 1, 0, 1)^2 \setminus \{(0, 0)\}$ (which are called small steps), how many walks are there from the origin (0, 0) to the $(i, j)$ using a steps taken from $SS$ . To shady this problem, we called the numbers $g_{j,k}$ is a generating series: $C(x, y, f) = \sum_{k \in S} E_{k} g_{k} g_{k}^{-1} g_{k}^{-1} f_{k}^{-1}$ . The classification of different models according to the algebraic and differential nature of the function $G(x, y, r_i)$ is done by using the recurrence relation $E_{k,j} = \sum_{k \in S} E_{k} g_{k} g_{k} g_{k}^{-1} g_{k}^{-1} f_{k}^{-1}$ . Nuch holds for all different models. Inderever, for each $(a, j) \in Z^2$ we have $E_k^{-1} e_{k} \sum_{k \in S} E_{k} g_{k} g_{k} g_{k}^{-1} g_{k} g_{k}^{-1}$ . Now, if we sum the left-hand sides of the previous equation over all $(a, b) \in S$ , we can inpuly the recurrence equation. This is to following equation: $E(x, y, f) = \sum_{k \in S} \sum_{k \in S} E_{k} g_{k} g_{k}^{-1} g_{k} g_{k}^{-1}$ . Now, if we sum the left-hand sides of the previous equation over all $(a, b) \in S$ , we can implify the recurrence equation. This is to following equation: $E(x, y, f) = \sum_{k \in S} \sum_{k \in S} g_{k} g_{k}$		The package como_warks		
the differences $y_i = p_i - p_{i-1} \in \mathbb{Z}^2$ is $1 \le n$ are its steps. Given a set $S \in \mathbb{Z}^2$ whose elements we call valid steps, we can consider valids in the lattice plane that only have steps in such subset, i.e., walks where $s_i \in S$ for all $1 \le i \le n$ . We call the scalarly and valids a model of value. The following combinatorial problem was considered • ( <i>i</i> , <i>j</i> ) using a steps taken from $S^2$ To sharp the problem, we collect the numbers $g_{i,k}$ in a generating series: $G(x_i, r) = \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} g_{i,k} a^{i,k} f^{i,k}$ . The classification of different models according to the algebra can different and are the function $G(x, y, r)$ is done by using the recurrence relation $g_{i,k} = \sum_{m \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} g_{i,k} a^{i,k} a^{i,k} a^{i,k} a^{i,k} a^{i,k}$ . The classification of different models. Moreover, for each $(a, b) \in \mathbb{Z}^2$ we have $\mu(x_j) \in \mathcal{L}_{i,k} = \sum_{m \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} g_{i,k} a^{i,k} a^{i,k} a^{i,k} a^{i,k} a^{i,k} a^{i,k}$ . Now, if we sum the left-hand aides of the previous equation over all $(a, b) \in S$ , we can insimply the right-hand aides using the recurrence equation. This is to following equation: $G(x, y, r) = \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} g_{k,k} a^{i,k} a^{i,k} a^{i,k} a^{i,k} a^{i,k}$ . Now, if we sum the left-hand aides of the previous equation over all $(a, b) \in S$ , we can insimply the right-hand aide using the recurrence equation. This is to be following equation: $G(x, y, r) = \frac{1}{1 - (x_i, y_i)}$ , where $S(x, y)$ is the so-called attep polynomial of the model S and its dised atter all solutions are polynomial and atter of the solution at the solution of the solution of the solution of a solution.		1. Introduction to walks in the plane		
Given a set $C \in Z^2$ whose elements we call value at the consider walks in the lattice plane that only have steps in such subset, i.e., walks where $s_i \in S$ for all $1 \leq n$ . We call the resulting set of walks. The following combinatorial problem was considered • Walks on the plane; given as $4 \leq C - 1$ , $0, 1 > 1$ { $(0, 0)$ [ which are called small steps), how many walks are there from the origin $(0, 0)$ to the $(i, j)$ using a steps taken from $S^2$ . To study this problem, we collect the numbers $g_{i,j,k}$ in a generating series: $G(x, y, r) = \sum_{(p,d) = m} g_{i,j,k} x^i y^j r^k$ . The classification of different models according to the algebraic and differential nature of the function $G(x, y, r)$ is done by using the recurrence relation $B_{i,j,k} = \sum_{(p,d) = m} g_{i,j,k} x^{i,j,k} + m + m + m + m + m + m + m + m + m + $		A walk in the lattice $\mathbb{Z}^2$ is a sequence of points $(P_0,, P_n)$ . We say that $P_0$ is the origin of the walk a	and $P_n$ is its ending point, that $n$ is it	s length and that al
$x_i \in S$ for all $1 \leq x$ . We call the resulting set of walks a model of walks. The following combinatorial problem was considered: • Walks on the place, given as $d \leq S - (-1, 0, 1)^2 \setminus \{(0, 0)\}$ (which are called small steps), how many walks are there from the orign $(0, 0)$ to the $i(j, j)$ using $n$ steps taken from $S = S$ . To study this problem, we collect the numbers $g_{ij,k}$ in a generating series: $G(x, y, n) = \sum_{ij \neq k} \sum_{k=0}^{k} g_{ij,k} x^i y^k n^k$ . The classification of different models according to the algebraic and differential nature of the function $G(x, y, n)$ is done by using the recurrence relation $Eu_{k,k} = \sum_{ij \neq k} \sum_{k=0}^{k} B_{ij,k} x^{k+n} y^{k+k+1}$ which holds for all different models. Moreover, for each $(a, b) \in \mathbb{Z}$ we have $E^{ij} G(x, y, n) \in \mathbb{Z}$ we have $E^{ij} G(x, y, n) = \sum_{k \neq k} \sum_{k=0}^{k} E_{k,k} x^{k+n} y^{k+k+1}$ $= \sum_{k \neq k} \sum_{k=0}^{k} E_{k,k} E^{ij} E^{ij} x^{k+n}$ . Now, if we sum the left-hand sides of the previous equation over all $(a, b) \in S$ , we can simplify the right-hand side using the neurence equation. This let to the following equation: $G(x, y, n) = \frac{1}{1 - \frac{1}{1 - \frac{1}{2}} x_{k+1}$ , where $S(x, y)$ is the so-called thep polynomial of the model S and to its defined as follows:		the differences $s_i = P_i - P_{i-1} \in \mathbb{Z}^2$ for $1 \le n$ are its steps.		
The following combinatorial problem was considered: • White on the plane; given a set $S \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ (which are called annal steps), how many walks are there from the origin $(0, 0)$ to the $p(x_i)$ by using a tops states them $S = S = S = S = S = S = S = S = S = S $		Given a set $S \subset \mathbb{Z}^2$ whose elements we call valid steps, we can consider walks in the lattice plane that	at only have steps in such subset, i.e	., walks where
• Walks on the plane given a set $S \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ (which are called small adeps), how many walks are there from the origin $(0, 0)$ to the $t_{i,j}(1)$ using $n$ stops taken from $S^7$ . To study this problem, we collect the numbers $g_{i,j,n}$ in a generating series: $G(x, y, t) \equiv \sum_{k \neq l} \sum_{m \in \mathbb{N}} g_{i,j,m}^{k} x^{j} t^{k}$ . The classification of different models according to the algebraic and differential nature of the function $G(x, y, t)$ is done by using the recurrence relation $g_{i,k,n} = \sum_{m \in \mathbb{N}} g_{i,m}^{k} x^{k} - \sum_{m \in \mathbb{N}} g_{i,k,m}^{k} x^{k} - \sum_{m \in \mathbb{N}} g_{i,k,m}^{k} x^{k} + \sum_{m \in \mathbb{N}} g_{i,k$		$s_i \in S$ for all $1 \le i \le n$ . We call the resulting set of walks a model of walks.		
(i, j) using a steps taken from S? To sludy this problem, we collect the numbers $g_{ij,k}$ in a generating series: $G(\mathbf{x}, \mathbf{y}, \mathbf{n}) = \sum_{i,j,k} \sum_{k=1}^{N} g_{ij,k} x^{ij} t^k$ . The classification of different models according to the algebraic and different nature of the function $G(\mathbf{x}, \mathbf{y}, \mathbf{n})$ is done by using the recurrence relation $\mathcal{B}(\mu_i = \sum_{k=1}^{N} \sum_{k=1}^{N} \partial_{ik} u_{ij} + \lambda_{ikm-1} + \lambda_{ikm-1} + \lambda_{ikm} + \lambda_{ikm$		The following combinatorial problem was considered:		
To shady this problem, we collect the numbers $g_{ij,k}$ in a generaling series: $G(x,y,t) = \sum_{ij,k,l} \sum_{max} g_{ij,k} x^{ij} y^{ik}.$ The classification of different models according to the algebraic and differential nature of the function $G(x,y,t)$ is done by using the recurrence relation $g_{ij,k,l} = \sum_{ij,k,k,l} g_{ij,k,l} x^{ik} y^{ik} g_{ij,k-1}$ which holds for all different models. Moreover, for each $(a, b) \in \mathbb{Z}^2$ being $B_{ij,k,l} x^{ik} y^{ik} g_{ij,k-1}$ which holds for all different models. Moreover, for each $(a, b) \in \mathbb{Z}^2$ being $B_{ij,k,l} x^{ik} g_{ij,k-1}$ which holds for all different models is different models. Moreover, for each $(a, b) \in \mathbb{Z}^2$ being $B_{ij,k,l} x^{ik} g_{ij,k-1}$ . Now, if we sum the lith-hand sides of the previous equation over all $(a, b) \in \mathbb{S}$ , we can simplify the right-hand side using the recurrence equation. This is to the following equation: $G(x,y,t) = \frac{1}{1 - \frac{1}{$			many walks are there from the origi	n $(0,0)$ to the point
$G(\mathbf{x}, y, t) = \sum_{k \neq k} \sum_{m \in \mathbb{N}} (\mathbf{x}_k)^{k \neq k} d^k \mathbf{x}_k$ The classification of different models according to the algebraic and differential nature of the function $G(\mathbf{x}, y, t)$ is done by using the recurrence relation $\frac{B(\mathbf{x})^2}{B(\mathbf{x})^2} \sum_{k \neq k} B(\mathbf{x}) - \frac{B(\mathbf{x})^2}{B(\mathbf{x})^2} \sum_{k \neq k} B(\mathbf{x}) - \frac{B(\mathbf{x}$		(i,j) using n steps taken from S?		
The classification of different models according to the algebraic and differential nature of the function $G(x, y, n)$ is done by using the recurrence relation $\begin{aligned} & \mathcal{K}_{U,R} = \sum_{k \in \mathcal{K}} \sum_{k \in \mathcal{K}} h_{k} u_{k} u_{k}$				
$\begin{split} & \mathcal{B}(j=\sum_{k\in \mathcal{K}} \mathcal{B}) \ \text{obs}_{k\in \mathcal{K}}(k) = \sum_{k\in \mathcal{K}} \mathcal{B} = \mathcal{B}(j=k) + \mathcal{B}$		$G(x, y, t) = \sum_{i,j \in \mathbb{Z}} \sum_{n \in \mathbb{N}} g_{i,j,n} x^i y^i t^n$ .		
which holds for all different models. Moreover, for each $(a, b) \in \mathbb{Z}^3$ , we have $\mathcal{U}^{a,g}^{b,g}(x, y, r) = \sum_{i,j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} N_{i,k} x^{k+a} y^{i+b} y^{i+b} x^{i+1}$ $\mathcal{U}^{a,g}^{b,g}(x, y, r) = \sum_{i,j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} N_{i,j} x^{k+a} y^{i+b} y^{i+d}$ . Now, if we sum the left-hand sides of the previous equation over all $(a, b) \in \mathbb{S}$ , we can simplify the right-hand side using the recurrence equation. This let to the following equation: $G(x, y, r) = \frac{1}{1 - nS(x, y)}$ , where $S(x, y)$ is the so-called step polynomial of the model $\mathbb{S}$ and its of defined as follows:		The classification of different models according to the algebraic and differential nature of the function G	(x, y, t) is done by using the recurre	ence relation
which holds for all different models. Moreover, for each $(a, b) \in \mathbb{Z}^3$ , we have $\mathcal{U}^{a,g}^{b,g}(x, y, r) = \sum_{i,j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} N_{i,k} x^{k+a} y^{i+b} y^{i+b} x^{i+1}$ $\mathcal{U}^{a,g}^{b,g}(x, y, r) = \sum_{i,j \in \mathbb{Z}} \sum_{k \in \mathbb{N}} N_{i,j} x^{k+a} y^{i+b} y^{i+d}$ . Now, if we sum the left-hand sides of the previous equation over all $(a, b) \in \mathbb{S}$ , we can simplify the right-hand side using the recurrence equation. This let to the following equation: $G(x, y, r) = \frac{1}{1 - nS(x, y)}$ , where $S(x, y)$ is the so-called step polynomial of the model $\mathbb{S}$ and its of defined as follows:		$g_{ij,n} = \sum g_{(i-a),(j-b),(n-1)}$		
$\begin{split} & xc^* q^* G(x,y,n) &= \sum_{k \mid d \in \mathcal{X}} \sum_{k \in \mathcal{M}} K u_k x^{k+T q^* d p^* d n^*} \\ &= \sum_{k \mid d \in \mathcal{X}} \sum_{k \in \mathcal{N}} K u_k u_k^{k+T q^* d p^* d n^*} \\ &\text{Now, if we sum the left-hand sides of the previous equation over all (a, b) \in \mathcal{S}, we can simplify the right-hand side using the recurrence equation. This let to the following equation: G(x,y,n) = \frac{1}{1 - (x,y_n)}, \\ \text{where } S(x,y) \text{ is the so-called step polynomial of the model S and it is defined as follows: } \end{split}$		(a,b)6S		
Now, if we sum the left-hand sides of the previous equation over all (a, b) $\in S$ , we can simplify the right-hand side using the recurrence equation. This let to the following equation: $G(x,y,t) = \frac{1}{1-nS(x,y)},$ where $S(x,y)$ is the so-called step polynomial of the model $S$ and its defined as follows:		$tx^a y^b G(x, y, t) = \sum_{i,i \in \mathbb{Z}} \sum_{n \in \mathbb{N}} g_{i,i,n} x^{i+a} y^{j+b} t^n$	+1	
to the following equation: $G(x, y, t) = \frac{1}{1 - L(x, y)}$ , where $S(x, y)$ is the so-called step polynomial of the model $S$ and it is defined as follows:		$= \sum_{i,j \in \mathbb{Z}} \sum_{n \ge 1} g_{(i-a),(j-b),(n+1)}$	ciyit".	
$G(x, y, t) = \frac{1}{1 - tS(x, y)}$ , where $S(x, y)$ is the so-called step polynomial of the model $S$ and it is defined as follows:		Now, if we sum the left-hand sides of the previous equation over all $(a, b) \in S$ , we can simplify the rig	ht-hand side using the recurrence e	quation. This leads
where $S(x, y)$ is the so-called step polynomial of the model S and it is defined as follows:		to the following equation:		
where $S(x, y)$ is the so-called step polynomial of the model S and it is defined as follows:		$G(x, y, t) = \frac{1}{1 - tS(x, y)},$		
		where $S(x, y)$ is the so-called step polynomial of the model S and it is defined as follows:		
	<b>C</b>	e package comb walks		

How to use it  $_{\circ\circ\circ}$ 

 $\underset{\circ\circ\circ}{\text{Conclusions}}$ 

# What can we do with comb\_walks

How to use it  $_{\circ\circ\circ}$ 

What to do with it 0 = 0 = 0 = 0

Conclusions

#### Main user data structure

#### WalkModel

- Each instance of the class represents a model of walks.
- The constructor inputs the set of valid steps.

In [2]: WalkModel((1,0),(0,1),(-1,-1))

Out[2]: Walk Model with steps: ((1, 0), 1), ((0, 1), 1), ((-1, -1), 1)

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What to do with it 0 = 0 = 0

Conclusions

#### Main user data structure

#### WalkModel

- Each instance of the class represents a model of walks.
- The constructor inputs the set of valid steps.
- Possible to use letters for the steps: N, S, E, W, etc.

In [3]: WalkModel(N,E,SW)

Out[3]: Walk Model with steps: ((0, 1), 1), ((1, 0), 1), ((-1, -1), 1)

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What to do with it

Conclusions

# Plotting methods



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Conclusions

# Plotting methods



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Conclusions

## Methods for geometric objects

#### Kernel function

```
The function K in \mathbb{P}^2:
```

In [13]: WalkModel(N,NE,W,SW).kernel()

Out[13]: 
$$(-t) x^2 y^2 + (-t) x y^2 z + x y z^2 + (-t) y z^3 + (-t) z^4$$

How to use it

Conclusions

## Methods for geometric objects

#### Kernel function

```
The function K in \mathbb{P} \times \mathbb{P}:
```

In [14]: WalkModel(N,NE,W,SW).kernel('P')

Out[14]:  $(-t) x_0^2 y_0^2 + (-t) x_0 x_1 y_0^2 + x_0 x_1 y_0 y_1 + (-t) x_1^2 y_0 y_1 + (-t) x_1^2 y_1^2$ 

How to use it  $_{\circ\circ\circ}$ 

Conclusions

#### Methods for geometric objects

#### Kernel function

The function K in  $\mathbb{P} \times \mathbb{P}$ :

#### The automorphism au

Map that changes first the y coordinate and then the x coordinate:

```
In [20]: WalkModel(N,NE,W,SW).tau('P')
```

Walks in the Quarter Plane  $_{\circ\circ\circ\circ\circ\circ}$ 

How to use it  $_{\circ\circ\circ}$ 

Conclusions

# Methods for geometric objects

# The function $b_i := \tau(F_i) - F_i$

Dut[23]: 
$$\frac{\left(\frac{-1}{t}\right)x^2y + \left(\frac{-1}{t}\right)xy + 2x + 1}{-x^2y - xy}$$

In [24]: WalkModel(N,NE,W,SW).b(2)  
Out[24]: 
$$\frac{x^2y^2 + xy^2 - 1}{-xy - y}$$

Walks in the Quarter Plane  $_{\circ\circ\circ\circ\circ\circ}$ 

How to use it  $_{\circ\circ\circ}$ 

Conclusions

# Telescoping functions over the curve

#### Poles of rational functions on the kernel curve

In [28]:	<pre>model = WalkModel(NE,S,W) model.poles(model.b(2))</pre>			
Out[28]:	[(0:1,1:0), (1:0,0:1), (0:1,0:1)]			



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What to do with it

Conclusions

# Telescoping functions over the curve

#### Poles of rational functions on the kernel curve

	<pre>model = WalkModel(NE,S,W) model.poles(model.b(2))</pre>			
Out[28]:	[(0:1,1:0), (1:0,0:1), (0:1,0:1)]			

#### Telescoping functions over the curve

Compute L and g such that  $L \cdot b = \tau(g) - g$ :

01

Out[31]: True

How to use it  $_{\circ\circ\circ}$ 

What to do with it

Conclusions

# Telescoping functions over the curve

#### Poles of rational functions on the kernel curve

In [28]:	<pre>model = WalkModel(NE,S,W) model.poles(model.b(2))</pre>			
Out[28]:	[(0:1,1:0), (1:0,0:1), (0:1,0:1)]			

#### Telescoping functions over the curve

Compute L and g such that  $L \cdot b = \tau(g) - g$ . In this case:

$$\delta b_2 = (\tau - 1) \left( t \frac{x_0^2 y_0 + x_1^2 y_1}{x_0 x_1 y_0} \right)$$

How to use it  $_{\circ\circ\circ}$ 

What to do with it 00000000

Conclusions

# Built-in models

- FiniteGroup: all models where  $\mathit{ord}(\tau) < \infty$ .
- EllipticC: models with infinite group and elliptic kernel.
- NonEllipticC: other models.

Conclusions

# Built-in models

- FiniteGroup: all models where  $\mathit{ord}(\tau) < \infty$ .
- EllipticC: models with infinite group and elliptic kernel.
- NonEllipticC: other models.

## Exhaustive website

The functionality of the package has been applied to all those models. The results can be found in the webpage

https://discretewalks.gitlabpages.inria.fr/comb\_walks/

Walks	in	the	Quarter	Plane

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Conclusions

# The website

#### https://discretewalks.gitlabpages.inria.fr/comb\_walks/



How to use it  $_{\circ\circ\circ}$ 

What to do with it 00000000

Conclusions

# The website

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Conclusions

# The website

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Conclusions

## The website

#### https://discretewalks.gitlabpages.inria.fr/comb\_walks/



Walks	in	the	Quarter	Plane

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What to do with it 00000000

Conclusions

# The website

#### https://discretewalks.gitlabpages.inria.fr/comb\_walks/



How to use it

What to do with it

Conclusions

# Conclusions

How to use it

What to do with it

Conclusions

# Conclusions

#### Achievements

- A framework to classify models of walks in the quarter plane
- Telescoping method over elliptic kernel functions
- Visualization of the results over classical examples

#### Future work

- Develop a constructive method for obtaining algebraic and differential equations
- Study the performance and complexity of the current code
- Adapt the current implementation to wider use

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# Thank you!

Results and documentation:

- https://discretewalks.gitlabpages.inria.fr/comb\_walks/
- https://discretewalks.gitlabpages.inria.fr/comb\_walks/docs/

GitLab repository:

• https://gitlab.inria.fr/discretewalks/comb\_walks