Doctoral Program Computational Mathematics Numerical Analysis and Symbolic Computation

FWF Der Wissenschaftsfonds.

DD-Finite Functions

Working beyond holonomic

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MCA - Symbolic Computation (July 2021)

D-finite functions: the holonomic world

Basic notation

Throughout this talk we consider:

- $\mathbb{K}$: a **computable** field contained in $\mathbb{C}$.
- $\mathbb{K}[[x]]$: ring of formal power series over $\mathbb{K}$.
- $'$ is the standard derivation w.r.t. x:

$$\left(\sum_{n\geq 0} c_n x^n\right)' = \sum_{n\geq 0} (c_n x^n)' = \sum_{n\geq 0} (n+1) c_{n+1} x^n.$$

Links to package

Package dd_functions

All results presented in this talk are included in the SageMath package dd_functions.

- **Repository:**
  https://github.com/Antonio-JP/dd_functions
- **Documentation:**
  https://antonio-jp.github.io/dd_functions/
- **Demo:**
  https://mybinder.org/v2/gh/Antonio-JP/dd_functions.git/master?filepath=dd_functions_demo.ipynb

D-finite functions

Definition

Let $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is D-finite if there exists $d \in \mathbb{N}$ and polynomials $p_0(x), \ldots, p_d(x) \in \mathbb{K}[x]$ (not all zero) such that:

$$p_d(x)f^{(d)}(x) + \ldots + p_0(x)f(x) = 0.$$

Examples

Many special functions are D-finite:

- Exponential functions: e^x.
- Trigonometric functions: $\sin(x)$, $\cos(x)$.
- Logarithm function: $\log(x+1)$.
- Bessel functions: $J_n(x)$.
- Hypergeometric functions: ${}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} ; x\right)$.
- Airy functions: $Ai(x), Bi(x)$.
- Combinatorial generating functions: $F(x), C(x), \ldots$

Closure properties

$f(x), g(x)$ D-finite of order d_1, d_2.
$a(x)$ algebraic over $\mathbb{K}(x)$ of degree p.

Property	**Function**	**Order bound**
Addition	$f(x) + g(x)$	$d_1 + d_2$
Product	$f(x)g(x)$	$d_1 d_2$
Differentiation	$f'(x)$	d_1
Integration	$\int f(x)$	$d_1 + 1$
Be Algebraic	$a(x)$	p

Working with D-finite functions

There are several implementations of D-finite functions:

- *mgfun*: Maple package, by F. Chyzak and B. Salvy
- *HolonomicFunctions*: Mathematica package, by C. Koutschan
- *ore_algebra*: Sage package, by M. Kauers et al.

Differentially definable functions: extending the class

Non-D-finite examples

There are power series that **are not** D-finite:

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

- $\wp$-Weierstrass function.
- Gamma function: $f(x) = \Gamma(x+1)$.
- Partition Generating Function: $f(x) = \sum_{n\geq 0} p(n)x^n$.

DD-finite functions

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$$p_d(x)f^{(d)}(x) + \ldots + p_0(x)f(x) = 0.$$

D-finite: **NO**

- Double exponential: $f(x) = e^{e^x}$.
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

DD-finite functions

Definition

Let $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is DD-finite if there exists $d \in \mathbb{N}$ and D-finite functions $r_0(x), \ldots, r_d(x)$ (not all zero) such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

DD-finite: YES

- Double exponential: $f(x) = e^{e^x} \rightarrow f'(x) - e^x f(x) = 0$
- Tangent: $\tan(x) = \frac{\sin(x)}{\cos(x)} \rightarrow \cos^2(x)\tan''(x) - 2\tan(x) = 0.$

Differentially definable functions

Definition

Let $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is DD-finite if there exists $d \in \mathbb{N}$ and D-finite functions $r_0(x), \ldots, r_d(x)$ (not all zero) such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

Differentially definable functions

Definition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring and $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ is differentially definable over R if there exists $d \in \mathbb{N}$ and elements in R $r_0(x), \ldots, r_d(x)$ (not all zero) such that:

$$r_d(x)f^{(d)}(x) + \ldots + r_0(x)f(x) = 0.$$

We denote the set of all diff. definable functions over R by $\mathrm{D}(R)$.

- D-finite functions: $\mathrm{D}(\mathbb{K}[x])$.
- DD-finite functions: $\mathrm{D}(\mathrm{D}(\mathbb{K}[x])) = \mathrm{D}^2(\mathbb{K}[x])$.

Characterization via Linear Algebra

Theorem

The following are equivalent:

$f(x)$ is differentially definable over R ($f(x) \in \mathrm{D}(R)$)

$$\Updownarrow$$

The F-vector space $\langle f(x), f'(x), f''(x), \dots \rangle$ has finite dimension.

- $R \subset K[[x]]$ is a differential subring
- F is its field of fractions.

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Closure properties

$f(x), g(x)$ in $\mathrm{D}(R)$ of order d_1, d_2.
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Proof for addition:

$$\begin{aligned}\langle (f+g)^{(n)} \;:\; n\in\mathbb{N}\rangle_F &= \langle f^{(n)}+g^{(n)} \;:\; n\in\mathbb{N}\rangle_F \\ &\subset \langle f^{(n)} \;:\; n\in\mathbb{N}\rangle_F + \langle g^{(n)} \;:\; n\in\mathbb{N}\rangle_F\end{aligned}$$

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D^n-finite functions: iterating the process

D^n-finite functions

Remark

$$R \subset \mathbb{K}[[x]] \text{ diff. ring} \Rightarrow \mathrm{D}(R) \subset \mathbb{K}[[x]] \text{ diff. ring}$$

$$\downarrow$$

Iterate the process

D^n-finite functions

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$$R \subset \mathbb{K}[[x]] \text{ diff. ring} \Rightarrow D(R) \subset \mathbb{K}[[x]] \text{ diff. ring}$$

$$\downarrow$$

Iterate the process

D^n-finite functions

D^n-finite functions are the *n*th iteration over $\mathbb{K}[x]$, i.e., $D^n(\mathbb{K}[x])$.

$$\mathbb{K}[x] \subset D(\mathbb{K}[x]) \subset D^2(\mathbb{K}[x]) \subset \ldots \subset D^n(\mathbb{K}[x]) \subset \ldots$$

New Properties

$f(x) \in D^n(\mathbb{K}[x])$ of order d_1.
$g(x) \in D^m(\mathbb{K}[x])$ of order d_2.
$a(x)$ algebraic over $D^m(\mathbb{K}[x])$ of degree p.

Property	**Function**	**Is in**	**Order bound**
Composition	$f \circ g$	$D^{n+m}(\mathbb{K}[x])$	d_1
Alg. subs.	$f \circ a$	$D^{n+m}(\mathbb{K}[x])$	pd_1

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- Then $f(a(x))$ is in $D^{n+m+1}(\mathbb{K}[x])$.

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- Then $f(a(x))$ is in $\mathrm{D}^{n+m+1}(\mathbb{K}[x])$.

$D^n \subsetneq D^{n+1}$: Iterated exponentials

$$K[x] \subsetneq D(K[x]) \subset D^2(K[x]) \subset \dots \subset D^n(K[x]) \subset \dots$$

$$e^x \notin K[x]$$

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Iterated Exponentials

- $e_0(x) = 1$,
- $\hat{e}_n(x) = \int_0^x e_n(t)dt$,
- $e_{n+1}(x) = \exp(\hat{e}_n(x))$.

$D^n \subsetneq D^{n+1}$: Iterated exponentials

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Iterated Exponentials

- $e_0(x) = 1,$
- $\hat{e}_n(x) = \int_0^x e_n(t)dt,$
- $e_{n+1}(x) = \exp(\hat{e}_n(x)).$

$$\begin{cases} e_{n+1}(x) \in D^{n+1}(K[x]) \\ e_{n+1}(x) \notin D^n(K[x]) \end{cases}$$

Diff. Algebraic functions

Definition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring and $f(x) \in \mathbb{K}[[x]]$. We say that $f(x)$ differentially algebraic over R if there is $n \in \mathbb{N}$ and $P(y_0, \ldots, y_n) \in R[y_0, \ldots, y_n]$ such that

$$P(f(x), f'(x), \ldots, f^{(n)}(x)) = 0.$$

We denote by $\mathrm{DA}(R)$ the set of all differentially algebraic functions over R.

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Diff. definable $\mathrm{D}(R)$	Diff. algebraic $\mathrm{DA}(R)$
$\downarrow$	$\downarrow$
Linear diff. equations	Polynomial diff. equations

Inclusion into Diff. Algebraic

- $\mathrm{D}(R) \subset \mathrm{DA}(R)$.
- $R \subset S \Rightarrow \mathrm{DA}(R) \subset \mathrm{DA}(S)$.
- $\mathrm{DA}(\mathbb{K}[x]) = \mathrm{DA}(\mathbb{K})$.

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Proposition

Let $R \subset \mathbb{K}[[x]]$ be a differential ring. Then $\mathrm{DA}(\mathrm{D}(R)) = \mathrm{DA}(R)$.

The proof is constructive.

Inclusion into Diff. Algebraic

- $\mathrm{D}(R) \subset \mathrm{DA}(R)$.
- $R \subset S \Rightarrow \mathrm{DA}(R) \subset \mathrm{DA}(S)$.
- $\mathrm{DA}(\mathbb{K}[x]) = \mathrm{DA}(\mathbb{K})$.
- **Proposition:** $\mathrm{DA}(\mathrm{D}^n(R)) = \mathrm{DA}(\mathrm{D}^{n-1}(R))$.

Theorem

For all $n \in \mathbb{N}$, if $f(x) \in \mathrm{D}^n(\mathbb{K}[x])$, then $f(x) \in \mathrm{DA}(\mathbb{K})$.

Inclusion into Diff. Algebraic

- $D(R) \subset DA(R)$.
- $R \subset S \Rightarrow DA(R) \subset DA(S)$.
- $DA(\mathbb{K}[x]) = DA(\mathbb{K})$.
- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R))$.
- **Theorem:** $D^n(\mathbb{K}[x]) \subset DA(\mathbb{K})$.

Example: double exponential

$$\exp(\exp(x) - 1) \longrightarrow f'(x) - \exp(x)f(x) = 0$$
$$\downarrow$$
$$f''(x)f(x) - f'(x)^2 - f'(x)f(x) = 0$$

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- **Proposition:** $DA(D^n(R)) = DA(D^{n-1}(R))$.
- **Theorem:** $D^n(\mathbb{K}[x]) \subset DA(\mathbb{K})$.

Example: tangent

$$\tan(x) \longrightarrow \cos(x)^2 f''(x) - 2f(x) = 0$$

$$\downarrow$$

$$-2f^{(5)}(x)f''(x)^2 f(x) + 12f^{(4)}(x)f'''(x)f''(x)f(x) - 6f^{(4)}(x)f''(x)^2 f'(x) - 12f'''(x)^3 f(x) + 12f'''(x)^2 f''(x)f'(x) - 4f'''(x)f''(x)^3 - 8f'''(x)f''(x)^2 f(x) + 8f''(x)^3 f'(x) = 0$$

Reverse inclusion

Is the other inclusion true? Can we have $\mathrm{DA}(\mathbb{K}[x]) = \mathrm{D}^{\infty}(\mathbb{K}[x])$?

Reverse inclusion

Is the other inclusion true? Can we have $\text{DA}(\mathbb{K}[x]) = \text{D}^\infty(\mathbb{K}[x])$?

For some diff. algebraic functions, we can find an $n \in \mathbb{N}$:

Riccati differential equation

Let $y(x)$ be a solution to the Riccati differential equation

$$y'(x) = c(x)y(x)^2 + b(x)y(x) + a(x),$$

where $a(x), b(x) \in \text{D}^n(\mathbb{K}[x])$ and $c(x) \in \text{D}^{n-1}(\mathbb{K}[x])$.
Then $y(x) \in \text{D}^{n+2}(\mathbb{K}[x])$.

Reverse inclusion

Is the other inclusion true? Can we have $\mathrm{DA}(\mathbb{K}[x]) = \mathrm{D}^\infty(\mathbb{K}[x])$?

But that is not always the case

Theorem (Noordman, Top, van der Put)

Let $y(x)$ be a solution to the differential equation

$$y'(x) = y(x)^3 - y(x)^2.$$

Then, there is no $n \in \mathbb{N}$ with $y(x) \in \mathrm{D}^n(\mathbb{K}[x])$.

Conclusions

Achievements

- Extension of the holonomic framework.
- Running implementation of closure properties.
- Relation to differentially algebraic functions.

Future work

- Fast computation of truncation of D^n-finite functions.
- Development of certified numerical evaluations.
- Combinatorial meaning of the induced sequences.
- Multivariate DD-finite functions.

Thank you!

Contact webpage:

- https://www.dk-compmath.jku.at/people/antonio
- https://www.risc.jku.at/home/ajpastor

Code available:

- https://github.com/Antonio-JP/dd_functions