On an autoconvolution problem

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- Motivation
- Equation
- Identifiability

2 Properties of the operator

3 Regularization

- Regularization algorithm
- Choice of the regularization parameter
- Results for artificial data

well-known: real, kernel-free autoconvolution problem

$$F_{\mathbb{R}}x = y, \qquad (1)$$

$$\int_{-\infty}^{s} x(s-t)x(t)dt = y(s), \qquad (2)$$

.....

$x(t)\in \mathbb{R} ext{ for } 0\leq t\leq 1, \ y(s)\in \mathbb{R} ext{ for } 0\leq s\leq 1 ext{ or } 0\leq s\leq 2.$

new: complex valued functions and nontrivial kernel

problem is provided by Max-Born-Institute for Nonlinear Optics and Short Time Spectroscopy, Berlin

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$\begin{array}{l} \text{SD-SPIDER} = \\ \underline{S}elf - \underline{D}efraction \ \underline{S}pectral \ \underline{P}hase \ \underline{I}nterferometry \ for \ \underline{D}irect \ \underline{E}lectric-field \\ \underline{R}econstruction \end{array}$



spectrograph measures Fourier-transformed signals

k-vector diagramme



problem: self-diffracted pulse is a convolution

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~

resulting equation

$$Fx = y$$
 (3)

$$F[x](s) = \int_{0}^{s} k(s,t)x(t)x(s-t)dt = y(s)$$
 (4)

$0 \leq t \leq 1$, $0 \leq s \leq 2$

with complex valued kernel $k(s,t) = \frac{\mu_0 cL}{2} \frac{s}{n(s)} \underline{\chi}^{(3)}(s,t) \overline{E}^{cw} e^{i(\Delta k_{\xi}\xi + \Delta k_{\eta}\eta + \Delta k_{\zeta}\frac{L}{2})} sinc(\Delta k_{\zeta}\frac{L}{2})$

> fundamental pulse: $x(t) = |x(t)|e^{iarphi_x(t)} \in \mathbb{C}$ SD-pulse: $y(s) = |y(s)|e^{iarphi_y(s)} \in \mathbb{C}$

to be reconstructed: $arphi_{\mathsf{x}}(t) = arphi_0 + \int_{-\infty}^{ au} {{{\mathbb{G}} D}(\hat{t})} d\hat{t}$.

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at first only φ_y and |x| available Does |y| have to be measured too?

Yes.

|y| carries significant information about $arphi_x$ this has been shown in analytical and numerical examples

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example of fundamental and convolved pulse, k(s,t) := 1



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$$F_{x} = y, \qquad F: L^{2}[0,1] \mapsto L^{2}[0,2]$$
 (5)

continuous

• Fréchet-derivative

$$[F'(x_0)h](s) = \int_0^s (k(s,t) + k(s,s-t))x_0(s-t)h(t)dt$$

- in general non-compact
- Fréchet-derivative always compact
- everywhere locally ill-posed

<u>Def.</u>: We define an operator of type (5) to be locally ill-posed in x_0 if, for arbitrarily small $\rho > 0$ there exists a sequence $\{x_n\} \subset B_{\rho}(x_0)$ satisfying the condition

$F(x_n) \to F(x_0)$ in Y as $n \to \infty$, but $x_n \nrightarrow x_0$ in X.

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$$F(x_n) \rightarrow F(x_0)$$
 in Y as $n \rightarrow \infty$, but $x_n \not\rightarrow x_0$ in X.

$$\begin{split} 0 &= [F(x_1)](s) - [F(x_2)](s) \\ &= \int_0^s k(s,t)(x_1(s-t) - x_2(s-t))(x_1(t) + x_2(t))dt + \\ &+ \int_0^s k(s,t)x_1(s-t)x_2(t)dt - \int_0^s k(s,t)x_2(s-t)x_1(t)dt. \end{split}$$

is zero if $x_1 = x_2$ or $x_1 = -x_2$. These are most likely the only two solutions.

Since $x_1 = -x_2$ means $x_1 = |x_2|e^{i(\varphi_{22} - \pi)}$ both solutions are equivalent for this problem.

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$$||\underline{y} - \underline{F}(\underline{x}_k) - \underline{F}'(\underline{x}_k)\underline{z}||_2^2 + \alpha_k ||\underline{L}\underline{z}||_2^2$$
(6)

with $\underline{L}\underline{z} = \underline{z}''$ as approximation of the second derivative

\Rightarrow iteration procedure

 $\underline{x}_{k+1} = \underline{x}_k + \gamma(\underline{F}'(\underline{x}_k)^* \underline{F}'(\underline{x}_k) + \alpha_k \underline{L}^* \underline{L})^{-1} \underline{F}'(\underline{x}_k)^* (\underline{y}^{\delta} - \underline{F}(\underline{x}_k))$ (7) starting value $\underline{x}_0 := |\underline{x}|^{\delta} e^{i\underline{0}}$

main stopping criteria $||\underline{y}^{\delta} - \underline{F}(\underline{x}_{k+1})||_2 \ge q||\underline{y}^{\delta} - \underline{F}(\underline{x}_k)||_2$, 0 < q < 1, for example q = 0.9999

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L-curve method and quasi-optimality failed, no information $||y - y^{\delta}|| < \delta$ for discrepancy principle

instead: using knowledge of the measured absolute values $|\underline{x}|^{\delta}$

calculating solutions $x^*(lpha_\ell)$ for a series of $lpha_\ell$, e.g. $lpha_\ell = lpha_0 \cdot q_lpha^\ell$, $\ell = 0, 1, \ldots, \ell_{max}$

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Introduction Regularization algorithm Properties of the operator Choice of the regularization parameter Regularization Results for artificial data

fundamental pulse used to create artificial data



Introduction Regularization algorithm Properties of the operator Regularization arameter Regularization Regularization parameter

x 10⁻²⁸ absolute value solution, a=0.60092 1.2 - - - original pulse Spectr. Pow. Dens. [x(i)] 8.0 4.0 2.0 2.0 starting phase reconstructed pulse 250 350 400 300 500 550 600 shase (arg(x(t))) -10 -15 -20 300 350 450 500 550 400 600 Frequency (THz)

reconstruction for $\delta = 0.1\%$

Introduction Regularization algorithm Properties of the operator Regularization arameter Regularization Regularization parameter

reconstruction for $\delta = 1\%$



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Introduction Regularization algorithm Properties of the operator Choice of the regularization parameter Regularization Results for artificial data

reconstruction for $\delta=5\%$



Introduction Regularization algorithm Properties of the operator Choice of the regularization parameter Regularization Results for artificial data

reconstruction for $\delta = 0\%$



Introduction Regularization algorithm Properties of the operator Regularization algorithm Regularization parameter

reconstruction for $\delta=1\%$



Introduction Regularization algorithm Properties of the operator Choice of the regularization parameter Regularization Results for artificial data

reconstruction for $\delta = 5\%$



Introduction	Regularization algorithm
Properties of the operator	Choice of the regularization parameter
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Thank you for your attention