On an autoconvolution problem

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well-known: real, kernel-free autoconvolution problem

$$F_{\mathbb{R}} x = y, \qquad (1)$$

$$\int_0^s x(s-t)x(t)dt = y(s), \qquad (2)$$

$x(t) \in \mathbb{R}$ for $0 \leq t \leq 1$, $y(s) \in \mathbb{R}$ for $0 \leq s \leq 1$ or $0 \leq s \leq 2$.

new: complex valued functions and nontrivial kernel

problem is provided by Max-Born-Institute for Nonlinear Optics and Short Time Spectroscopy, Berlin

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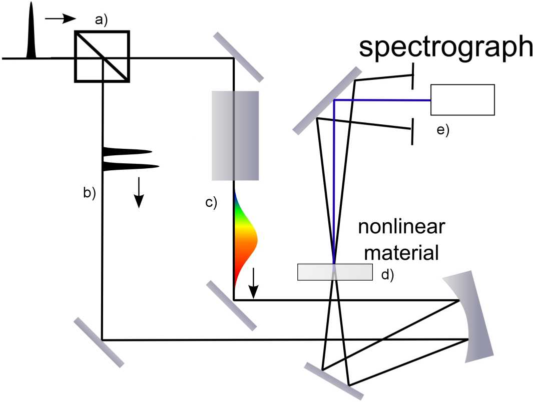
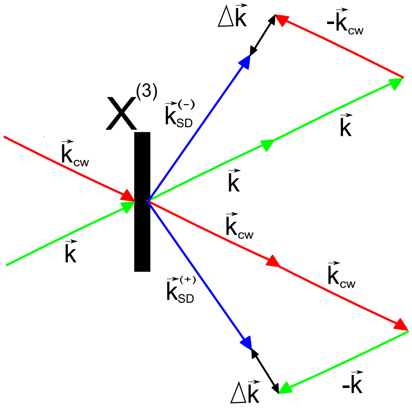
SD-SPIDER=
Self-Defraction Spectral Phase Interferometry for Direct Electric-field Reconstruction

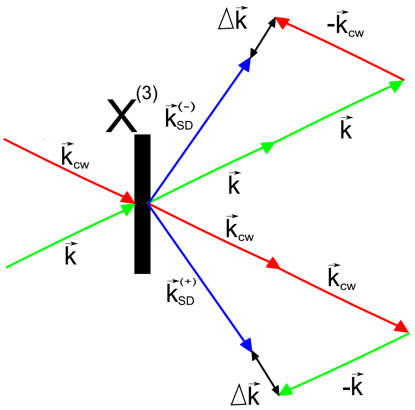


spectrograph measures Fourier-transformed signals

k-vector diagramme



problem: self-diffracted pulse is a convolution k-vector diagramme



problem: self-diffracted pulse is a convolution

resulting equation

$$Fx = y \tag{3}$$

$$F[x](s) = \int_0^s k(s,t)x(t)x(s-t)dt = y(s) \tag{4}$$

$0 \leq t \leq 1$, $0 \leq s \leq 2$

with complex valued kernel

$k(s,t) = \frac{\mu_0 cL}{2}\frac{s}{n(s)}\underline{\chi}^{(3)}(s,t)\overline{E}^{cw}\, e^{i(\Delta k_\xi \xi + \Delta k_\eta \eta + \Delta k_\zeta \frac{L}{2})} sinc(\Delta k_\zeta \frac{L}{2})$

fundamental pulse: $x(t) = |x(t)|e^{i\varphi_x(t)} \in \mathbb{C}$

SD-pulse: $y(s) = |y(s)|e^{i\varphi_y(s)} \in \mathbb{C}$

to be reconstructed: $\varphi_x(t) = \varphi_0 + \int_{-\infty}^{t} GD(\hat{t})d\hat{t}$

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Does $|y|$ have to be measured too?

Yes.

$|y|$ carries significant information about φ_x
this has been shown in analytical and numerical examples

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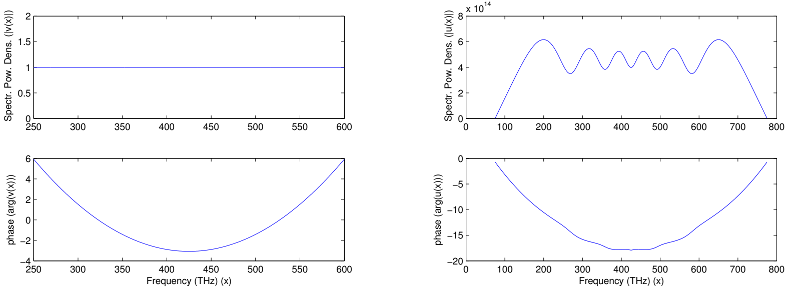
example of fundamental and convolved pulse, $k(s,t) := 1$



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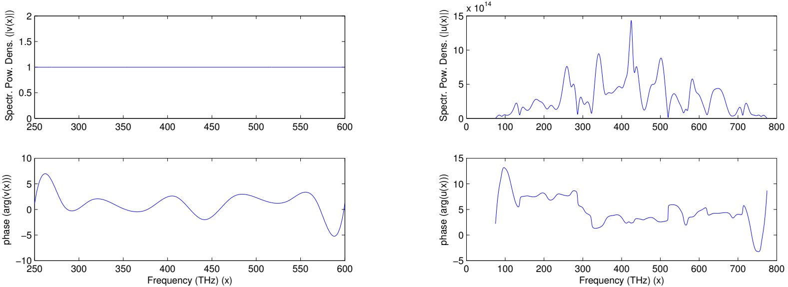

$\Rightarrow$ available data $|y|$, φ_y, $|x|$

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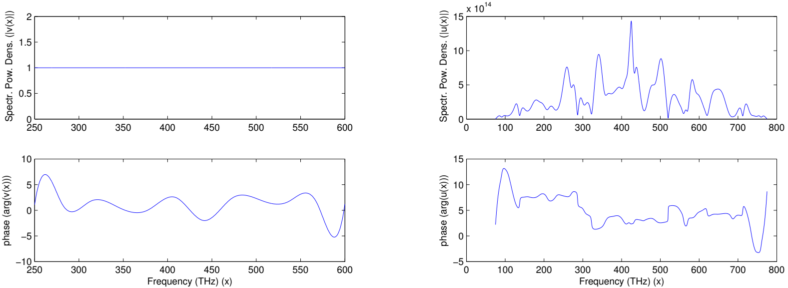

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$$Fx = y, \qquad F : L^2[0,1] \mapsto L^2[0,2] \tag{5}$$

- continuous
- Fréchet-derivative
  $$[F'(x_0)h](s) = \int_0^s (k(s,t) + k(s,s-t))x_0(s-t)h(t)dt$$
- in general non-compact
- Fréchet-derivative always compact
- everywhere locally ill-posed

Def.: We define an operator of type (5) to be locally ill-posed in x_0 if, for arbitrarily small $\rho > 0$ there exists a sequence $\{x_n\} \subset B_\rho(x_0)$ satisfying the condition

$$F(x_n) \to F(x_0) \text{ in } Y \text{ as } n \to \infty, \text{ but } x_n \nrightarrow x_0 \text{ in } X.$$

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Injectivity: Let $x_1,\ x_2 \in L^2[0,1]$. Then

$$
\begin{aligned}
0 &= [F(x_1)](s) - [F(x_2)](s) \\
&= \int_0^s k(s,t)(x_1(s-t) - x_2(s-t))(x_1(t) + x_2(t))dt + \\
&\qquad + \int_0^s k(s,t)x_1(s-t)x_2(t)dt - \int_0^s k(s,t)x_2(s-t)x_1(t)dt.
\end{aligned}
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is zero if $x_1 = x_2$ or $x_1 = -x_2$.
These are most likely the only two solutions.

Since $x_1 = -x_2$ means $x_1 = |x_2| e^{i(\varphi_{x_2} - \pi)}$ both solutions are equivalent for this problem.

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using Levenberg-Marquardt-type algorithm to minimize in each iteration the linearized problem

$$||\underline{y} - \underline{F}(\underline{x}_k) - \underline{F}'(\underline{x}_k)\underline{z}||_2^2 + \alpha_k||\underline{L}\,\underline{z}||_2^2 \qquad (6)$$

with $\underline{L}\,\underline{z} = \underline{z}''$ as approximation of the second derivative

$\Rightarrow$ iteration procedure

$$\underline{x}_{k+1} = \underline{x}_k + \gamma(\underline{F}'(\underline{x}_k)^*\underline{F}'(\underline{x}_k) + \alpha_k\underline{L}^*\underline{L})^{-1}\underline{F}'(\underline{x}_k)^*(\underline{y}^\delta - \underline{F}(\underline{x}_k)) \qquad (7)$$

starting value $\underline{x_0} := |\underline{x}|^\delta e^{i\underline{0}}$

main stopping criteria $||\underline{y}^\delta - \underline{F}(\underline{x}_{k+1})||_2 \geq q||\underline{y}^\delta - \underline{F}(\underline{x}_k)||_2$, $0 < q < 1$, for example $q = 0.9999$

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constant $\alpha_k := \alpha$ in all iterations to preserve smoothing properties

L-curve method and quasi-optimality failed, no information $||y - y^\delta|| < \delta$ for discrepancy principle

instead: using knowledge of the measured absolute values $|\underline{x}|^\delta$

calculating solutions $x^*(\alpha_\ell)$ for a series of α_ℓ, e.g. $\alpha_\ell = \alpha_0 \cdot q_\alpha^\ell$, $\ell = 0, 1, \ldots, \ell_{max}$

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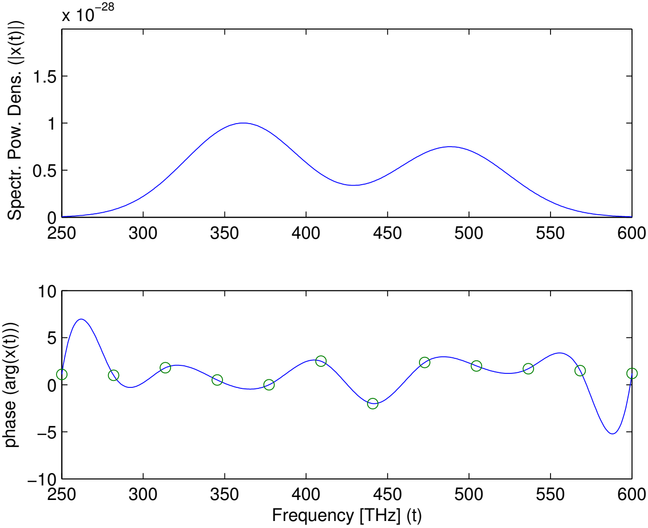
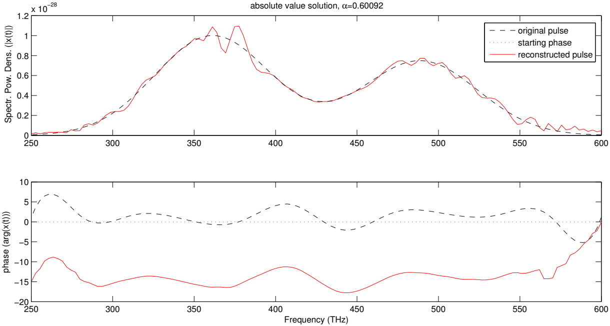
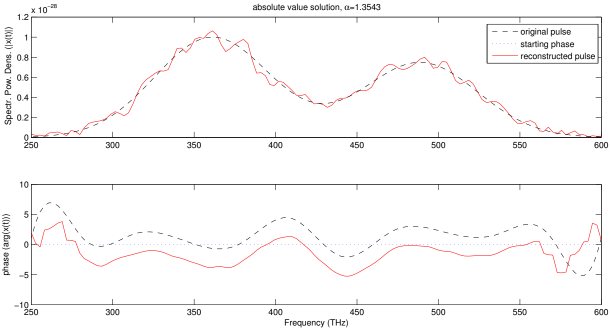
fundamental pulse used to create artificial data



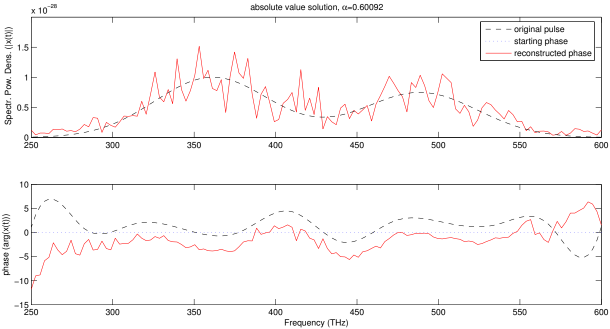
reconstruction for $\delta = 0.1\%$



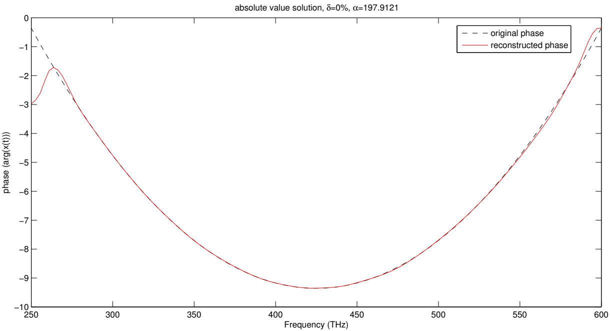
reconstruction for $\delta = 1\%$



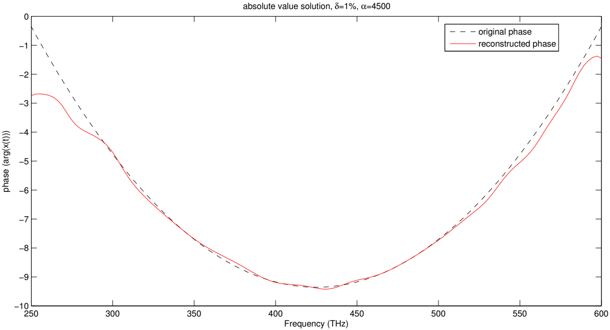
reconstruction for $\delta = 5\%$



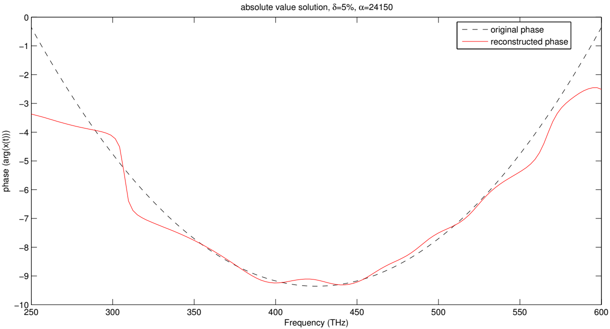
reconstruction for $\delta = 0\%$



reconstruction for $\delta = 1\%$



reconstruction for $\delta = 5\%$



Thank you for your attention