A Stochastic Convergence Analysis for Tikhonov-Regularization with Sparsity Constraints

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Overview

Starting question

Can we prove convergence (-rates) for Tikhonov-Regularization with sparsity-penalty if instead of $||y - y^\delta|| \leq \delta$ an explicit stochastic error model is used?

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- We study the solution of the linear ill-posed problem

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

  with $\mathbf{A} \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ where $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces.
- Basic deterministic model:

$$||\mathbf{A}\mathbf{x} - \mathbf{y}^\delta||^2 + \hat{\alpha}\Phi_{\mathbf{w},p}(\mathbf{x}) \to \min_{\mathbf{x}} \tag{1}$$

- Penalty $\Phi_{\mathbf{w},p}(\mathbf{x}) = \sum_{\lambda \in \Lambda} w_\lambda |\langle \mathbf{x}, \psi_\lambda \rangle|^p$ for an ONB $\{\psi_\lambda\}$

- computation requires discretization, done via projections

$$
\begin{aligned}
P_m &: \mathcal{Y} \to \mathbb{R}^m, \quad \mathbf{y} \mapsto y, \quad \text{e.g. point evaluation} \\
T_n &: \mathcal{X} \to \mathbb{R}^n, \quad x = T_n \mathbf{x} = \{\langle \mathbf{x}, \psi_i \rangle\}_{i=1,\dots,n} \\
T_n^* &: \mathbb{R}^n \to \mathcal{X}, \quad \mathbf{x}_n = \sum_{i=1}^{n} \langle \mathbf{x}, \psi_i \rangle \psi_i
\end{aligned}
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where $\{\psi_i\}_{i=1}^{\infty}$ is ONB in $\mathcal{X}$.

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- each component of y carries *stochastic* noise, $y^\sigma = y + \varepsilon$, $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_m)$.

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- each component of y carries *stochastic* noise, $y^\sigma = y + \varepsilon$, $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_m)$.
- Define $A := P_m \mathbf{A} T_n^*$, then we want to find x s.t.

$$Ax = y^\sigma \tag{2}$$

We use Bayes' formula

to calculate the solution. In this framework, every quantity is treated as a random variable in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

$$\pi_{post}(x|y^\sigma) = \frac{\pi_\varepsilon(y^\sigma|x)\pi_{pr}(x)}{\pi_{y^\sigma}(y^\sigma)}.$$

- $\pi_{post}(x|y^\sigma)$ posterior density
- $\pi_\varepsilon(y^\sigma|x)$ likelihood function
- $\pi_{pr}(x)$ prior distribution
- $\pi_{y^\sigma}(y^\sigma)$ data distribution (irrelevant)

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gaussian error model:

$$\pi_\varepsilon \propto exp(-\frac{1}{2\sigma^2}||Ax - y^\sigma||^2),$$

Besov spaces

- We are looking for sparse reconstructions w.r.t. a basis in $\mathcal{X}$
- our choice: Besov-Space $B^s_{p,q}(\mathbb{R}^d)$ prior (notation: $B^s_p(\mathbb{R}^d)$)
- Reasons:
  - "easy" characterization with coefficients of a wavelet expansion
  - sparsity-promoting properties
  - discretization invariance (Lassas, Saksman, Siltanen '09)

Definition of the wavelet expansion

Let $\phi \in C^{\tilde{s}}(\mathbb{R})$ be a scaling function and $\psi \in C^{\tilde{s}}(\mathbb{R})$ a compactly supported wavelet, $\tilde{s} > s$ and define

$$\phi_{j,k}(t) = 2^{\frac{j}{2}}\phi(2^j t - k),$$
$$\psi_{j,k}(t) = 2^{\frac{j}{2}}\psi(2^j t - k), \quad j,k \in \mathbb{Z}.$$

such that the collection $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ constitutes an orthonormal basis of $L_2(\mathbb{R})$ and for each $\mathbf{h} \in L_2(\mathbb{R})$ we have

$$\mathbf{h} = \sum_{k=-\infty}^{\infty} \langle \mathbf{h}, \phi_{0,k}\rangle \phi_{0,k} + \sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty} \langle \mathbf{h}, \psi_{j,k}\rangle \psi_{j,k}.$$

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assumption:
only finitely many nonzero scalar products on each scale

- with this, a d-dimensional basis can be constructed
- let $\{\psi_\lambda : \lambda \in \Lambda\}$ denote the set of all wavelets ψ, also including the scaling functions ϕ where Λ is an appropriate index set, possibly infinite
- set $|\lambda| = j$, then
- $\mathbf{x} \in B_p^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, $s < \tilde{s}$, if

$$||\mathbf{x}||_{B_p^s} := \left(\sum_{\lambda \in \Lambda} \underbrace{2^{\varsigma p |\lambda|}}_{w_\lambda} |\langle \mathbf{x}, \psi_\lambda \rangle|^p \right)^{1/p} < \infty$$

and $\varsigma = s + d(\frac{1}{2} - \frac{1}{p}) \geq 0$. We focus on $1 \leq p \leq 2$.

Besov-space random variables

Definition (adapted from Lassas/Saksman/Siltanen, 2009)

Let $1 \leq p < \infty$ and $s \in \mathbb{R}$. Let U be the random function

$$U(t) = \sum_{\lambda \in \Lambda} 2^{-\varsigma|\lambda|} X_\lambda^\alpha \psi_\lambda(t), \quad t \in \mathbb{R}^d,$$

where the coefficients $(X_\lambda^\alpha)_{\lambda \in \Lambda}$ are independent identically distributed real-valued random variables with probability density function

$$\pi_X(\tau) = c_p^\alpha \exp(-\frac{\alpha|\tau|^p}{2}), \quad c_p^\alpha = \left(\frac{\alpha}{2}\right)^{\frac{1}{p}} \frac{p}{2\Gamma(\frac{1}{p})}, \quad \tau \in \mathbb{R}.$$

Then we say U is distributed according to a B_p^s-prior, $U \propto \exp(-\frac{\alpha}{2}||U||_{B_p^s}^p)$.

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Theorem (adapted from Lassas/Saksman/Siltanen, 2009)

Let U be a B_p^s-random function defined as before, $2 < \alpha < \infty$ and take $r \in \mathbb{R}$. Then the following three conditions are equivalent:

(i) $||U||_{B_p^r(\mathbb{R}^d)} < \infty$ *almost surely,*

(ii) $\mathbb{E} \exp\left(||U||^p_{B_p^r(\mathbb{R}^d)}\right) < \infty$,

(iii) $r < s - \frac{d}{p}$.

same result as [LSS 2009], but here $\mathbb{R}^d$ instead of $\mathbb{T}^d$ considered

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  - consider discretization level $n < \infty$ fixed, finite index set Λ_n
  - Then

$$X_n(t) := \sum_{\lambda \in \Lambda_n} 2^{-\varsigma|\lambda|} X_\lambda^\alpha \psi_\lambda(t) \Rightarrow ||X_n||_{B_p^s}^p = \sum_{\lambda \in \Lambda_n} |X_\lambda^\alpha|^p < \infty$$

  - and $\mathbb{P}(||X_n||_{B_p^s} > \varrho) = \frac{\Gamma(\frac{n}{p}, \frac{\alpha \varrho^p}{2})}{\Gamma(\frac{n}{p})} \leq \frac{1}{\varrho} \sqrt[p]{\frac{2n}{\alpha p}}$
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- "infinite model" (MII)
  - forget about discretization level n
  - define $X(t)$ in $B_p^r(\mathbb{R}^d)$ with $s < r - \frac{d}{p}$, then
  - $\mathbb{E}(||X||_{B_p^s(\mathbb{R}^d)}) = \left(\frac{2}{\alpha p}\left(c_\lambda^1 + c_\lambda^2 \sum_{j=0}^\infty 2^{-j((r-s)p-d)}\right)\right)^{\frac{1}{p}} < \infty$
  - and $\mathbb{P}(||X||_{B_p^s} > \varrho) \leq \frac{1}{\varrho}\mathbb{E}(||X||_{B_p^s(\mathbb{R}^d)})$

Recall

$$\pi_{post}(x|y^\sigma) = \frac{\pi_{pr}(x)\pi_\varepsilon(y^\sigma|x)}{\pi_{y^\sigma}(y^\sigma)}.$$

$\pi_\varepsilon(y^\sigma|x)$ gaussian noise, $\pi_{pr}(x)$ Besov-Space prior
$\Rightarrow \pi_{post}(x|y^\sigma) \propto \exp(-\frac{1}{2\sigma^2}||Ax - y^\sigma||^2) \cdot \exp(-\frac{\alpha}{2}||T_n^* x||_{B_p^s}^p)$

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we are interested in the *maximum a-priori* solution

$$x_{\text{map}} = \underset{x\in\mathbb{R}^n}{\text{argmax}} \quad \pi_{post}(x|y^\sigma)$$

or equivalently

$$x_\alpha^{\text{map}} = \underset{x\in\mathbb{R}^n}{\text{argmin}} \quad ||Ax - y^\sigma||^2 + \alpha\sigma^2 ||T_n^* x||_{B_p^s(\mathbb{R}^d)}^p \tag{3}$$

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same functional as in deterministic case, but $||y - y^\sigma|| \leq \delta$ does not hold

- stochastic setting requires different measure for convergence
- convergence in expectation may be too strict
- instead, we use the *Ky Fan metric*

Definition

Let x_1 and x_2 be random variables in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a metric space (χ, d_χ). The distance between x_1 and x_2 in the *Ky Fan metric* is defined as

$$\rho_K(x_1, x_2) := \inf\{\epsilon > 0 : \mathbb{P}(d_\chi(x_1(\omega), x_2(\omega)) > \epsilon) < \epsilon\}.$$

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- allows combination of deterministic and stochastic quantities
- metric for convergence in probability

Ky Fan error estimate

Theorem (Neubauer, Pikkarainen, 2008)

Let y^σ *be a random variable with values in* $\mathbb{R}^m$. *Assume that the distribution of* y^σ *is* $\mathcal{N}(y, \sigma^2 I)$ *with* $\sigma > 0$. *Then it holds in* $(\mathbb{R}^m, ||\cdot||)$ *that*

$$\rho_K(y^\sigma, y) \leq \min\left\{1, \sqrt{2}\sigma\sqrt{m - \ln^-\left(\sigma^2 2\pi m^2 \left(\frac{e}{2}\right)^m\right)}\right\},$$

where $f^-(h) := \min\{0, f(h)\}$.

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in practice $\ln$-term mostly inactive, then

$$\rho_K(y^\sigma, y) \leq \min\left\{1, \sqrt{2}\sigma\sqrt{m}\right\},$$

c.f. $\mathbb{E}(||y^\sigma - y||) \approx \sigma\sqrt{m}$

Overview

Let $x^\dagger$ be the unique solution of the equation $Ax = y$ with minimum value of $\Phi(\cdot)$.

Theorem (adapted from Hofinger, '06)

Let $\alpha, \sigma > 0$, $1 \leq p \leq 2$ and $N(A) = 0$ for $p = 1$. Assume $w_\lambda \geq c > 0 \; \forall \lambda \in \Lambda$. Let x_α^{map} be the solution of (3). If $\alpha = \alpha(\sigma)$ is chosen such that $\hat{\alpha} = \alpha\sigma^2 \to 0$ and $\frac{|\ln \sigma|}{\alpha} \to 0$ as $\sigma \to 0$, then

$$\lim_{\sigma \to 0} \rho_K(x_\alpha^{map}, x^\dagger) = 0.$$

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$$\lim_{\sigma \to 0} \rho_K(x_\alpha^{map}, x^\dagger) = 0.$$

as long as $\sigma^2 2\pi m^2 \left(\frac{e}{2}\right)^m > 1$, then $\alpha \to \infty$ is sufficient

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$$\lim_{\sigma \to 0} \rho_K(x_\alpha^{\mathit{map}}, x^\dagger) = 0.$$

as long as $\sigma^2 2\pi m^2 \left(\frac{e}{2}\right)^m > 1$, then $\alpha \to \infty$ is sufficient

main idea for the proof: use Ky Fan metric and split $\Omega = \Omega_{\text{det}}(\sigma) \cup \Omega_{\text{unbound}}(\sigma)$

Overview

Deterministic Convergence Rate, Daubechies et al, 2004

Assume $\mathbf{A}$ fulfils, for all $h \in L^2$

$$A_l^2 \sum_\lambda 2^{-2|\lambda|\beta} |\langle h, \psi_\lambda \rangle|^2 \leq ||Ah||^2 \leq A_u^2 \sum_\lambda 2^{-2|\lambda|\beta} |\langle h, \psi_\lambda \rangle|^2. \quad (4)$$

and $||\mathbf{x}^\dagger||_{B_p^s} \leq \varrho$, $\varrho > 0$. Then

$$\sup\{||\mathbf{x}_\alpha^{\text{map}} - \mathbf{x}|| : \mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}, ||\mathbf{Ax} - \mathbf{y}|| \leq \delta, ||\mathbf{x}||_{B_p^s} \leq \varrho\} < C \left(\frac{\delta + \delta'}{A_l} \right)^{\frac{\varsigma}{\beta+\varsigma}} (\varrho + \varrho')^{\frac{\beta}{\beta+\varsigma}}$$

with $\delta' = (\delta^2 + \hat{\alpha}\varrho^p)^{\frac{1}{2}}$ and $\varrho' = (\varrho^p + \frac{\delta^2}{\hat{\alpha}})^{\frac{1}{p}}$.

Theorem

Let all previous assumptions hold. Set $a_m := \ln\left(\frac{2^m}{2\pi m^2}\right)$. *Then as* $\sigma \to 0$, x_α^{map} *converges with the parameter choice* $\alpha = \alpha(\sigma, \varrho, \beta, \varsigma, p, m, n)$ *fulfilling*

$$f(\alpha) := \min\left\{1, 2\left(\frac{\sqrt{2}}{A_l}\sigma\sqrt{a_m - 2\ln\sigma + \frac{\alpha\varrho^p}{2}}\right)^{\frac{\varsigma}{\beta+\varsigma}}\left(\left(\varrho^p + \frac{2}{\alpha}(a_m - 2\ln\sigma)\right)^{1/p}\right)^{\frac{\beta}{\beta+\varsigma}}\right\} - \frac{\Gamma(\frac{m}{2}, m)}{\Gamma(\frac{m}{2})} - \mathbb{P}(||x.||_{B_p^s} > \varrho) = 0$$

to the unique solution $x^\dagger$ *and*

$$\rho_K(x_\alpha^{map}, x^\dagger) = \mathcal{O}\left(\left(\sigma\sqrt{1 + |\ln\sigma| + \alpha\varrho^p}\right)^{\frac{\varsigma}{\beta+\varsigma}}\left(\left(\varrho^p + \frac{1 + |\ln\sigma|}{\alpha}\right)^{1/p}\right)^{\frac{\beta}{\beta+\varsigma}}\right).$$

where $\mathbb{P}(||x_n||_{B_p^s} > \varrho) = \frac{\Gamma(\frac{n}{p}, \frac{\alpha\varrho^p}{2})}{\Gamma(\frac{n}{p})}$ or $\mathbb{P}(||x||_{B_p^s} > \varrho) = \frac{\mathbb{E}||x||_{B_p^s}}{\varrho}$

Overview

- Introduction
- A Convergence Theorem
- Convergence Rates
- Numerical Examples

- We consider a convolution problem

$$[Ax](s) = [k * x](s) = \int_{\mathbb{R}^d} k(s-t)x(t)dt, \quad s \in \mathbb{R}^d \qquad (5)$$

- using a kernel

$$\widehat{k}(\xi) = \frac{c_{\kappa,\beta}}{(1+\kappa|\xi|^2)^{\beta/2}}, \quad \xi \in \mathbb{R}^d, \quad c_{\kappa,\beta} \text{ s.t. } ||\widehat{k}||_{L_2(\mathbb{R}^d)} < 1$$

- thus (4) is fulfilled with chosen β
- $p = 1$, $d = 1$

Iteration [Daubechies, De Mol, Defrise 2004]:

With $x_0 = 0$,

$$x_{k+1} = \mathcal{S}_{\mathbf{w},p}\left(x_k + A^*(y^\sigma - Ax_k)\right), \qquad k = 1, 2, \dots,$$

where $\mathcal{S}_{\mathbf{w},p}(h) := \sum_{\lambda\in\Lambda} S_{w_\lambda,p}(\langle h, \psi_\lambda\rangle)\psi_\lambda$ is defined component-wise ($p = 1$) via

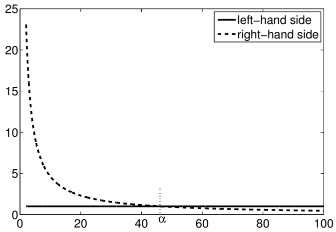
$$S_{\omega,1}(\xi) := \begin{cases} \xi - \frac{\omega}{2} & \text{if } \xi \geq \frac{\omega}{2} \\ 0 & \text{if } |\xi| < \frac{\omega}{2} \\ \xi + \frac{\omega}{2} & \text{if } \xi \leq -\frac{\omega}{2} \end{cases}.$$

converges since $||A|| < 1$

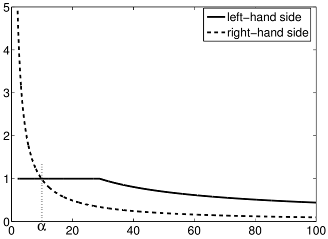
Parameter choice rule illustrated

$$\sigma = 0.01,\ m = 2500,\ \varsigma = 0.5,\ \beta = 1,\ \varrho = 2.16$$

model (MI), $s = 1$



model (MII), $s = 1$, $r = 2$



example of a solution

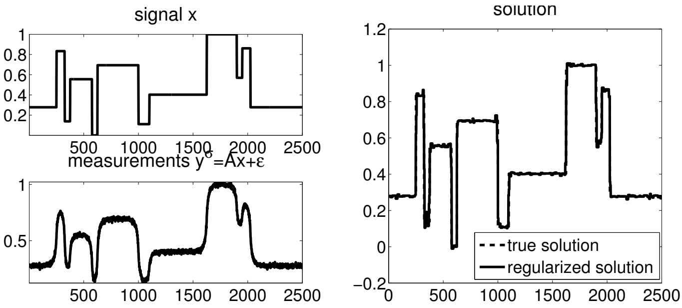


Figure : (MI), $\sigma = 0.01$, exact ϱ, $s = 1$, $\beta = 1$. $\alpha = 45.85$
$\Rightarrow \hat{\alpha} = \alpha\sigma^2 = 0.004585$

comparison of (MI) and (MII), m, n fixed, $\sigma \to 0$

all plots averaged over 20 individual simulations

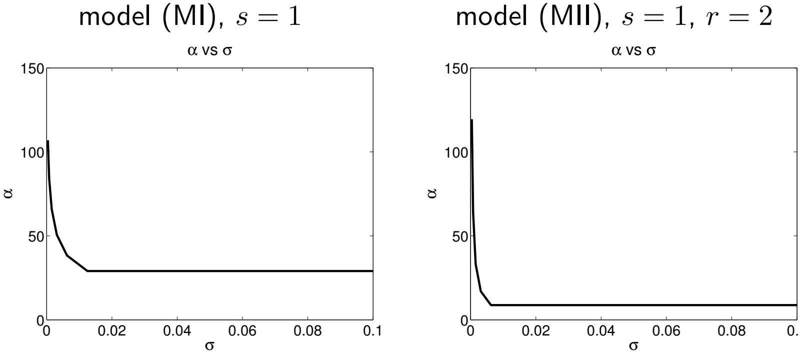


Figure : α plotted against σ, $n = m = 2500$, $\beta = 1$, exact ϱ

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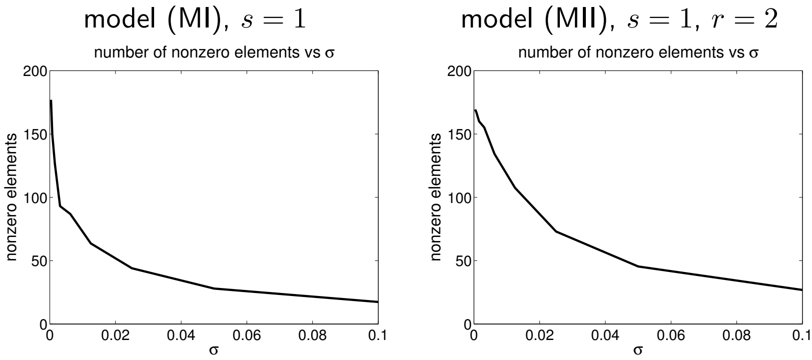


Figure : number of recovered nonzero coefficients plotted against σ, $n = m = 2500$, $\beta = 1$, exact ϱ

comparison of (MI) and (MII), m, n fixed, $\sigma \to 0$

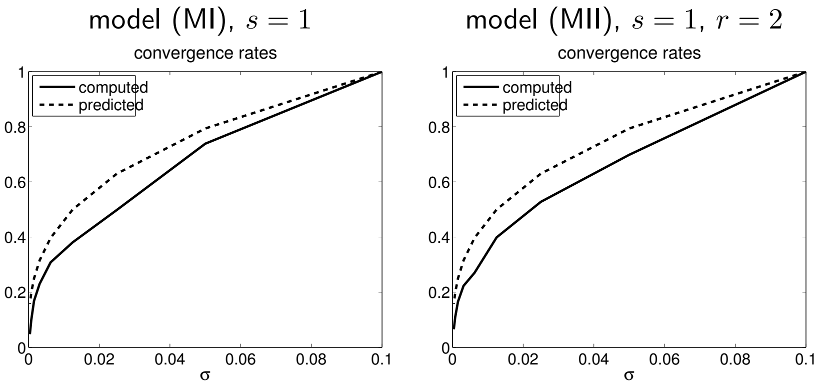


Figure : predicted and observed convergence rates plotted against σ, $n = m = 2500$, $\beta = 1$, exact ϱ

comparison of (MI) and (MII), σ fixed, m, n variable

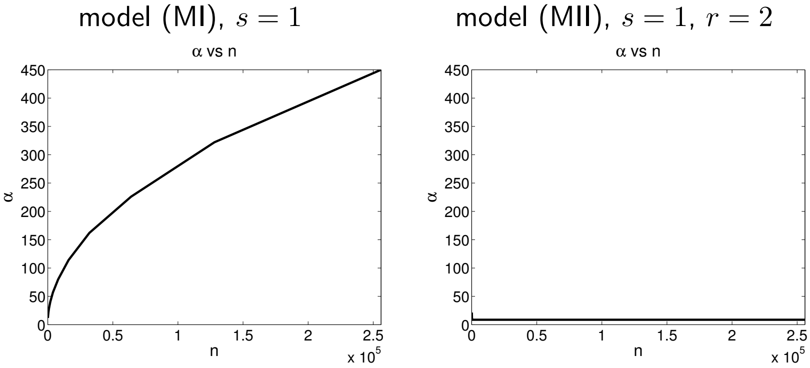
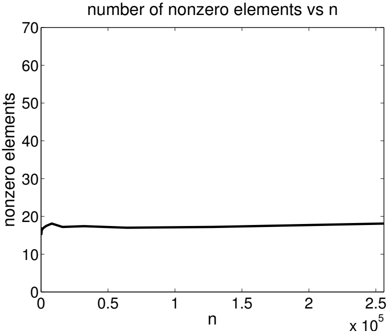


Figure : α plotted against n, $\sigma = 0.01$, $\beta = 1$, exact ϱ

comparison of (MI) and (MII), σ fixed, m, n variable

model (MI), $s = 1$



model (MII), $s = 1$, $r = 2$

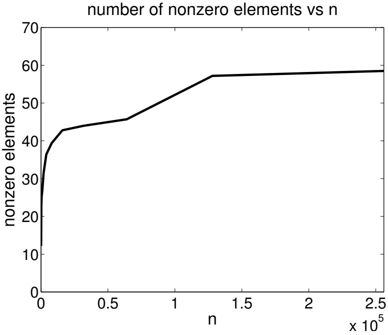


Figure : number of recovered nonzeros plotted against n, $\sigma = 0.01$, $\beta = 1$, exact ϱ

comparison of (MI) and (MII), σ fixed, m, n variable

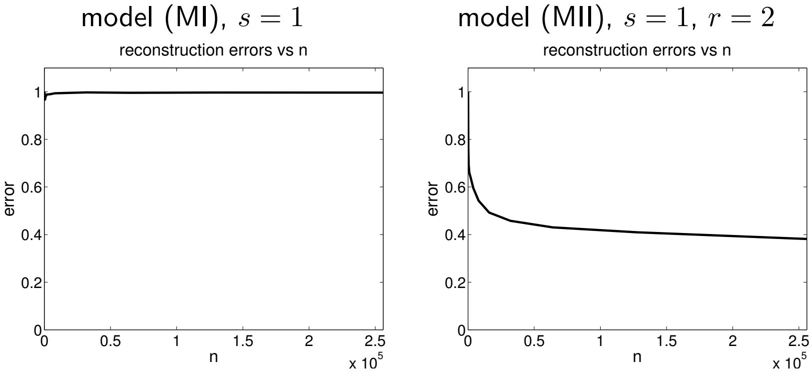


Figure : reconstruction error plotted against n, $\sigma = 0.01$, $\beta = 1$, exact ϱ

A 2D convolution example

$\sigma = 0.1$, $\beta = 1$, $\alpha = 130.5$, $\hat{\alpha} = 1.3$

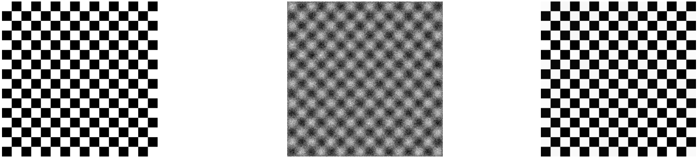


Figure : true solution - measurements - recovered solution

exactly the 68 original coefficients were reconstructed

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Thank you for attention! Are there questions?