Regularization of an autoconvolution problem in ultrashort laser pulse characterization

D. Gerth[a,b], B. Hofmann[b], S. Birkholz[c], S. Koke[c], G. Steinmeyer[c]

[a]Johannes Kepler University, Linz, Austria
[b]Chemnitz University of Technology, Germany
[c]Max Born Institute, Berlin, Germany

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

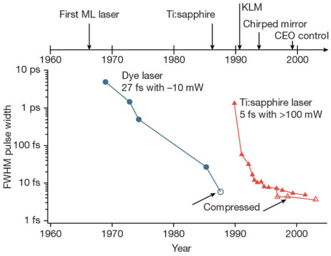
JKU
JOHANNES KEPLER UNIVERSITY LINZ

Overview

Motivation

Why study ultra-short laser pulses?
to create shorter, stronger pulses; to enhance optical systems; medicine, material processing, etc.

Problem: measurements limited by electronics (order 10^{-12}s)
Development of pulse durations:



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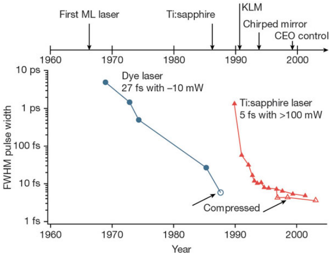
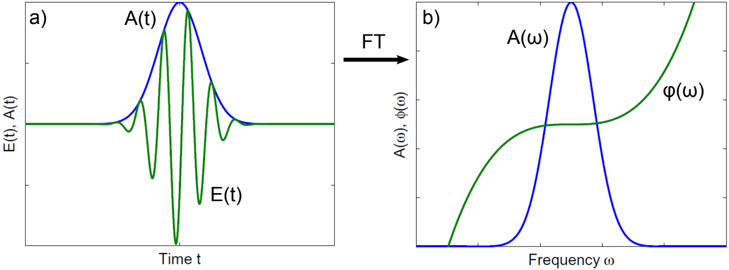


Solution: sample pulse by itself

Laser pulse representation

Time domain:
electric field $E(t)$, envelope $A(t)$, intensity $I(t) = |A(t)|^2$



Fourier domain:
amplitude $\mathcal{A}(\omega)$, phase $\varphi(\omega)$, spectrum $\mathcal{I}(\omega) = |\mathcal{A}(\omega)|^2$

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  Self-Defraction Spectral Phase Interferometry for Direct Electric-field Reconstruction

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- theory presented at "Conference on Lasers and Electro-Optics", 2010
- reasons for introduction: applicable for ultraviolet radiation, good signal strength because it uses third-order optical effects

basics of nonlinear optics

- Polarization $\tilde{P}$ caused by an electric field $\tilde{E}$,

$$\tilde{P}(t) = \epsilon_0[\chi^{(1)}\tilde{E}(t) + \chi^{(2)}\tilde{E}^2(t) + \chi^{(3)}\tilde{E}^3(t) + \dots]$$

  may act as source of electromagnetic radiation:

$$\nabla \times (\nabla \times E) + \frac{n^2}{c^2}\partial_t^2 E = -\mu_0 \partial_t^2 P_{\mathsf{NL}}(E)$$

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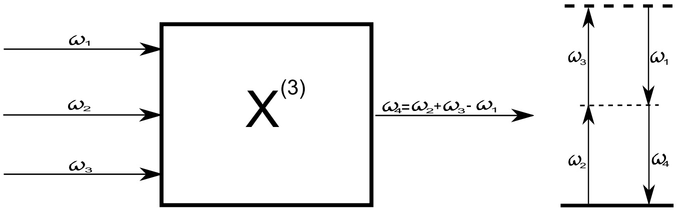
- Refraction index n and Kerr-effect:

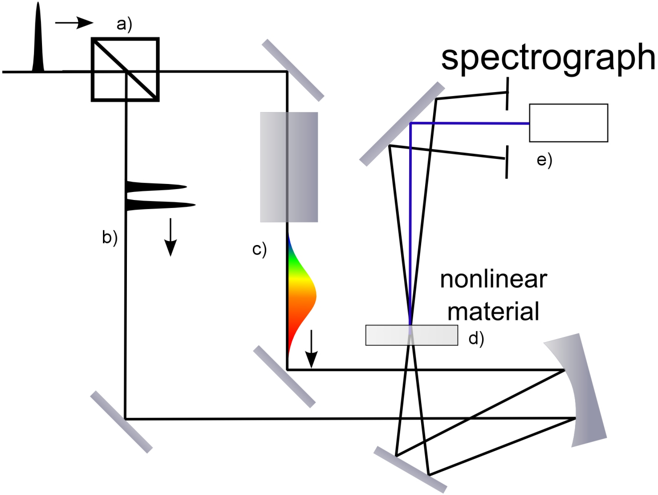
$$n(\omega) = n_0 + n_2|E(\omega)|^2,$$

  (each frequency is refracted slightly differently)

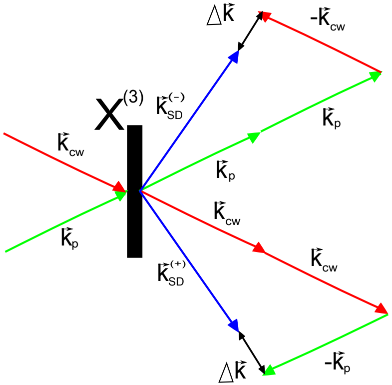
$\chi^{(3)}$-media allow a four-wave mixing process



Principle



k-vector-diagram:



$$\vec{\Delta}k(\omega_{SD}, \omega_\mathsf{p}, \omega_\mathsf{cw})$$
$$= -\vec{k}_{cw}(\omega_{cw}) + \vec{k}_p(\omega_\mathsf{p}) + \vec{k}_p(\omega_{SD} + \omega_{cw} - \omega_\mathsf{p}) - \vec{k}_{SD}(\omega_{SD}, \omega_{cw}, \omega_\mathsf{p}).$$

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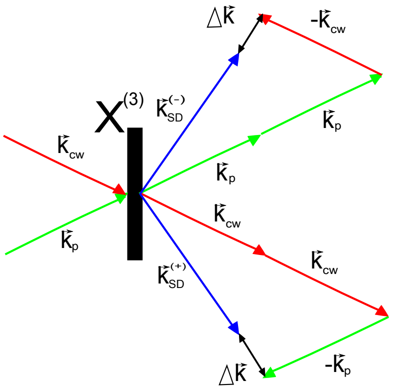

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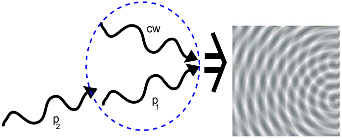
energy conservation $\omega_{\mathsf{p}} + \omega_{\mathsf{p}} = \omega_{SD} + \omega_{cw}$ still holds

The autoconvolution effect

- pulses considered as plane waves:



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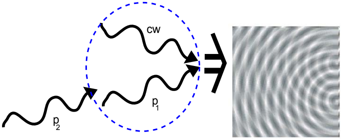

- interference pattern creates refractive index grating (Kerr-effect)

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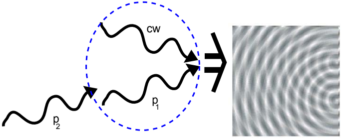

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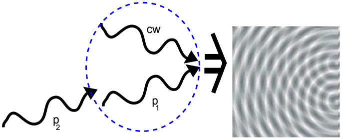

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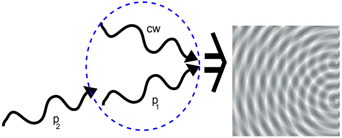

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- SD-signal is sum of all combinations
  $\mathcal{E}_p(\omega_\mathsf{p})\mathcal{E}_p(\omega_{SD}+\omega_{cw}-\omega_\mathsf{p})\mathcal{E}_{cw}$

Equation in physical formulation

$$\mathcal{E}_{SD}(\omega_{SD}) = \int\limits_{0}^{\omega_{SD}+\omega_{cw}} \mathcal{K}(\omega_{SD}, \omega_{\mathsf{p}})\mathcal{E}_p(\omega_{\mathsf{p}})\mathcal{E}_p(\omega_{SD} + \omega_{cw} - \omega_{\mathsf{p}})d\omega_{\mathsf{p}}$$

$\mathrm{supp}\,\mathcal{E}_p = [\omega_{\mathsf{p}}^l, \omega_{\mathsf{p}}^u]$, $\mathrm{supp}\,\mathcal{E}_{SD} = [2\omega_{\mathsf{p}}^l - \omega_{cw}, 2\omega_{\mathsf{p}}^u - \omega_{cw}]$,

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kernel

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$$\overline{\mathcal{E}}^{cw} e^{i(\Delta\vec{k}_\xi \xi + \Delta\vec{k}_\eta \eta + \Delta\vec{k}_\zeta \frac{L}{2})} sinc(\Delta\vec{k}_\zeta \frac{L}{2})$$

$\mathcal{K}$ continuous, complex valued

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unknown, so far neglected

mathematical formulation

- after transformation and renaming:

$$y(s) = F[x](s) = \int_0^s k(s,t)x(t)x(s-t)dt$$

$$y = F(x) \qquad 0 \leq t \leq 1, 0 \leq s \leq 2$$

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fundamental pulse: $x(t) = A(t)e^{i\varphi(t)}$

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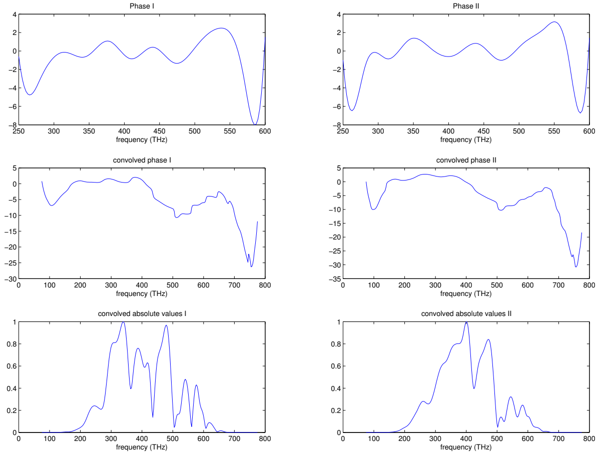
  available, possibly available, unknown
- $\varphi(t) = \varphi_0 + \int_{-\infty}^t GD(\tau)d\tau$

Does $B(s)$ provide important information?



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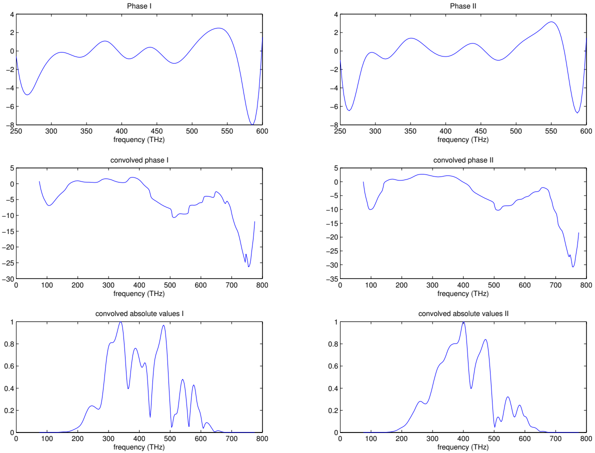

Yes, it does! Thus also $B(s)$ available as measurement.

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- no information about size of error δ available
- Statement of the problem:
  given A^δ, B^δ, ψ^δ and $k(s,t)$, find φ such that

$$B^\delta(s)e^{i\psi^\delta(s)} = \int_0^s k(s,t)A^\delta(t)e^{i\varphi(t)}A^\delta(s-t)e^{i\varphi(s-t)}dt$$

Overview

Ill-posedness

$$Fx = y, \qquad F : L^2[0,1] \mapsto L^2[0,2]$$

An operator F is called *ill-posed*, if it violates at least one of

Hadamard's conditions:

(a) for each given data y there exists a solution x

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(c) the solution depends continuously on the data

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(a) violated because $F(x) \in C_{\mathbb{C}}[0,2] \ \forall x \in L^2_{\mathbb{C}}[0,1]$

Injectivity

- for $k(s,t) \equiv 1$ and $k(s,t) = k(s)$: $F(x_1) = F(x_2)$ has two solutions $x_1 = x_2$ and $x_1 = -x_2$ by Titchmarsh's theorem

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- because of periodicity, $\varphi \equiv \varphi + 2\pi$

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- nonlinear operator requires local analysis

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Definition

We define an operator $\mathcal{F}$, $\mathcal{F}:\mathcal{X}\to\mathcal{Y}$ to be locally ill-posed in $x_0\in\mathcal{X}$ if, for arbitrarily small $\rho>0$ there exists a sequence $\{x_n\}\subset B_\rho(x_0)\subset X$ satisfying the condition

$$\mathcal{F}(x_n)\to\mathcal{F}(x_0) \text{ in } \mathcal{Y} \text{ as } n\to\infty, \text{ but } x_n\nrightarrow x_0 \text{ in } \mathcal{X}.$$

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Theorem (Gorenflo & Hofmann '94, adapted in Gerth '11)

The autoconvolution operator F is everywhere locally ill-posed.

$\Rightarrow$ (c) is violated too! Regularization is necessary.

Fréchet-derivative

The Fréchet-derivative of F in a point x_0 is given by

$$[F'(x_0)h](s) = \int_0^s (k(s,t) + k(s,s-t))x_0(s-t)h(t)dt$$

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although F is in general non-compact, F' is always compact!

Overview

- equation: $y(s) = \int\limits_0^s k(s,t)x(s-t)x(t)dt$
- $\operatorname{supp} x = [t_l, t_u]$, $\operatorname{supp} y = [2t_l - t_{cw}, 2t_u - t_{cw}]$

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- $\text{supp}\, x = [t_l, t_u]$, $\text{supp}\, y = [2t_l - t_{cw}, 2t_u - t_{cw}]$
- discretization using rectangular rule

$$y(s_m) = \sum_{j=1}^{N} k(s_m, t_j)x(s_m + t_{cw} - t_j)x(t_j)\Delta t$$

with $\Delta t = \frac{t_u - t_l}{N-1}$, $t_j = t_l + (j-1)\Delta t$, $s_m = 2t_j + (m-1)\Delta t$
$y_m := y(s_m)$, $x_n := x(t_n)$, $k_{m,n} := k(s_m, t_n)$

in matrix-form $\underline{y} = \underline{F}(\underline{x})\underline{x}$, with in matrix-form $\underline{y} = \underline{F}(\underline{x})\,\underline{x}$, with

$$\underline{y}/\Delta t = \underline{F}\,\underline{x}/\Delta t =$$

$$\begin{pmatrix}
k_{1,1}x_1 & 0 & \dots & 0 & 0 \\
k_{2,1}x_2 & k_{2,2}x_1 & \dots & 0 & 0 \\
 & \ddots & \ddots & & \vdots \\
k_{N-1,1}x_{N-1} & k_{N-1,2}x_{N-2} & \dots & k_{N-1,N-1}x_1 & 0 \\
k_{N,1}x_N & k_{N,2}x_{N-1} & \dots & k_{N,N-1}x_2 & k_{N,N}x_1 \\
0 & k_{N+1,1}x_N & \dots & k_{N+1,N-1}x_3 & k_{N+1,N-1}x_2 \\
\vdots & & \ddots & \ddots & \\
0 & 0 & \dots & k_{2N-2,N-1}x_N & k_{2N-2,N}x_{N-1} \\
0 & 0 & \dots & 0 & k_{2N-1,N}x_N
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Decomposition, with $\circ$ as element-by-element multiplication:
$\underline{F} = \underline{K} \circ \underline{X}$

analogously: Fréchet-derivative

$$[\underline{F'}(\underline{x_0})\underline{h}]_m = \sum_{j=1}^{N}(k(s_m,t_j)+k(s_m,s_m+t_{cw}-t_j))x_0(s_m+t_{cw}-t_j)h(t_j)\Delta t$$

analogously: Fréchet-derivative

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advantage: time-consuming calculation of the matrices $\underline{K}$ and $\underline{K'}$ has to be performed only once for each measurement setup

Overview

A Levenberg-Marquardt-Type approach

- we let the complete pulse x be unknown, whereas y is given
- Iteration rule:

$$\underline{x}^{\delta}_{(l+1)} := \underline{x}^{\delta}_{(l)} + \gamma \left(\underline{F}'(\underline{x}^{\delta}_{(l)})^* \underline{F}'(\underline{x}^{\delta}_{(l)}) + \alpha \underline{L}^* \underline{L} \right)^{-1} \underline{F}'(\underline{x}^{\delta}_{(l)})^* (\underline{y}^{\delta} - \underline{F}(\underline{x}^{\delta}_{(l)})$$

for $l = 0, \ldots, l^*$, aimed at minimizing

$$\|\underline{y}^{\delta} - \underline{F}(\underline{x}_{(l)}) - \underline{F}'(\underline{x}_{(l)})(\underline{x} - \underline{x}_{(l)})\|^2 + \alpha \|\underline{L}(\underline{x} - \underline{x}_{(l)})\|^2,$$

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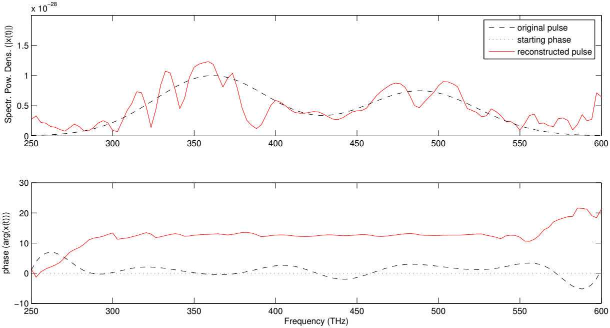
- Questions:
  - how to choose $\underline{x}_0$?
  - how to choose l^*?
  - how to choose α?

Choice of $\underline{x}_0 = A_0 e^{i\varphi_0}$

obviously, $A_0 := A^\delta$

first idea for phase: $\varphi_0(t) \equiv 0$

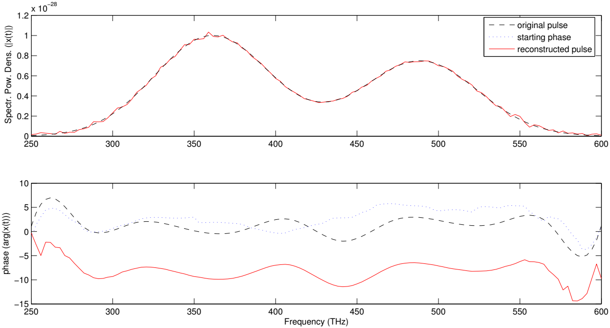


$(\delta = 0,\ \alpha = 0)$

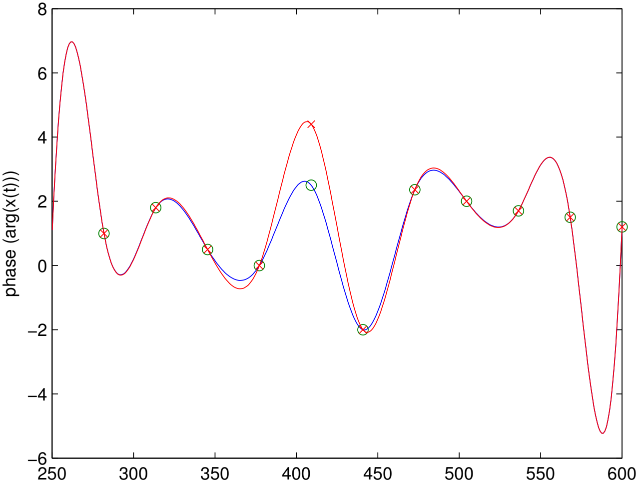
idea: calculate good guess. Observe

$$B^\delta(s)e^{i\psi^\delta(s)} = \int_0^s |k(s,t)| A^\delta(t) A^\delta(s-t) e^{i(\varphi(t)+\varphi(s-t)+\phi_{\text{kernel}})} dt$$

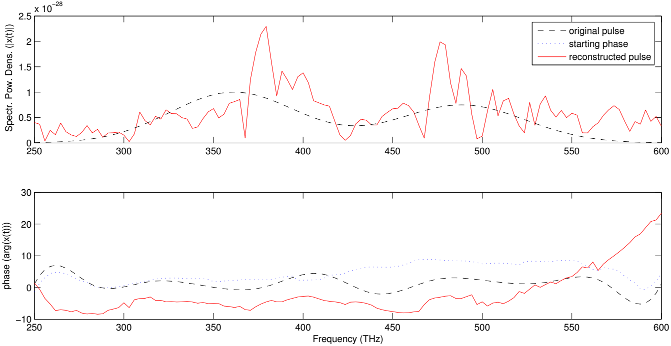
$\Rightarrow$ set $\varphi_0(t) = \frac{1}{2}(\mathcal{P}_{s\mapsto t}(\psi(s))) - \phi_{\text{kernel}}(s^*, t)$ for s^* fixed



problem for slightly changed fundamental phase



best result with kernel correction



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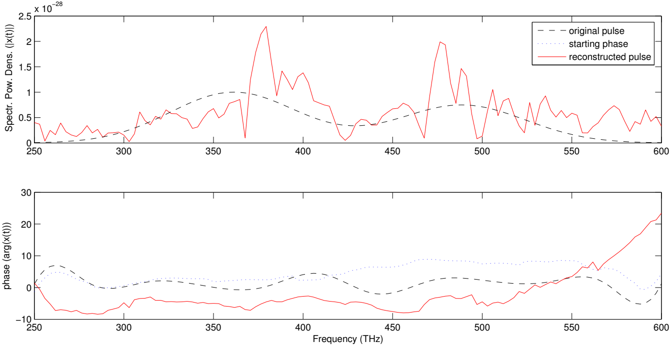

$\Rightarrow$ set starting phase to constant zero

When to stop the iteration?

An example iteration:

(l)	$\|\|\underline{F}(\underline{x}^{\delta}_{(l)}) - \underline{y}^{\delta}\|\|$	$\|\| \, \|\underline{x}^{\delta}_{(l)}\| - A^{\delta}\|\|$
1	9.5819e-01	0.5252
20	2.4115e-02	0.7916
40	2.0682e-02	0.7937
60	1.5369e-02	0.6077
100	1.3792e-03	0.1964
120	1.1022e-03	0.1701
140	9.4595e-04	0.1623
143	9.2340e-04	0.1622
144	9.1606e-04	0.1623
150	8.7480e-04	0.1632
250	3.1613e-04	0.2020

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$\Rightarrow$ choose l^* such that $|||\underline{x}^{\delta}_{(l)}| - \underline{A}^{\delta}||$ is minimal

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- no a-priori information $||y - y^\delta|| < \delta$ available, thus a-posteriori methods necessary

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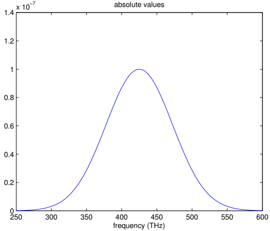
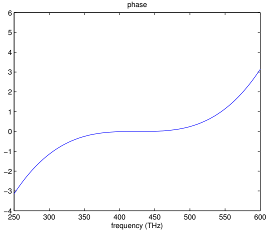
- L-curve not applicable, quasioptimality ($||x_{\alpha_{i+1}} - x_{\alpha_i}|| \to min$) failed
- instead, make use of A^δ again:
  choose α^* such that

$$|||\underline{x}^\delta_{\alpha^*}| - A^\delta|| = \min_n |||\underline{x}^\delta_{\alpha_n}| - A^\delta||$$

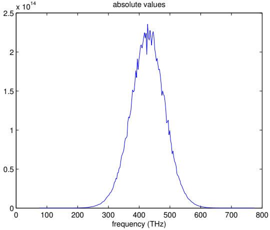
Overview

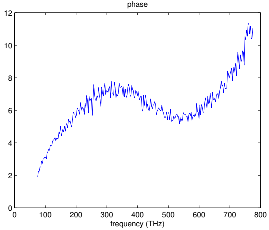
A very smooth fundamental pulse



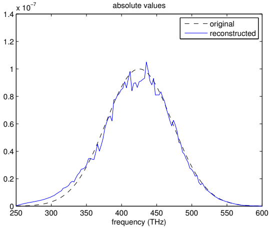


SD-pulse, 5% relative noise added

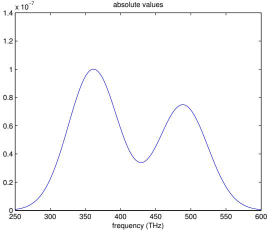


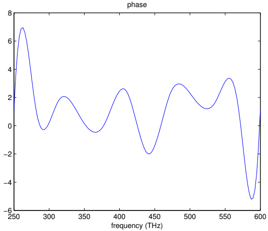


reconstruction, $\alpha = 5.86 \cdot 10^6$

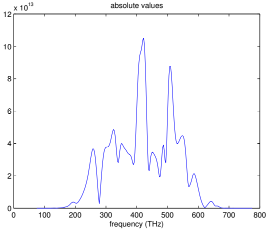


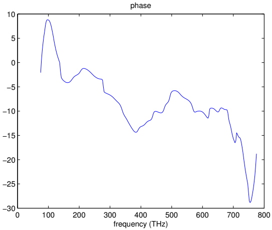
A more oscillating pulse



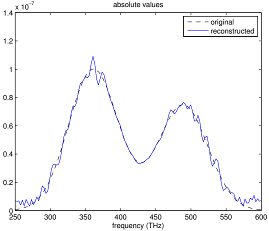


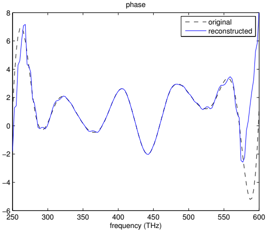
noise-free SD-pulse



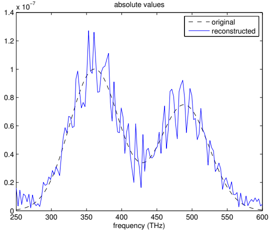


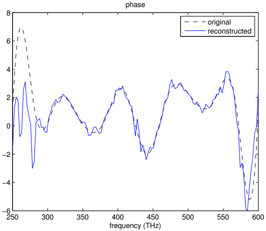
reconstruction, $\alpha = 2.17$





reconstruction, 1% relative noise in data





Real data situation

unfortunately, no results available. Main reasons:

- measurements without magnitudes
- unknown factor in model $\Rightarrow$ error in the model
- frequency domains of x and y do not match

- D. Gerth, B. Hofmann, S. Birkholz, S. Koke, and G. Steinmeyer *Regularization of an autoconvolution problem in ultrashort laser pulse characterization*, submitted
- D. Gerth, *Regularization of an autoconvolution problem occurring in measurements of ultra-short laser pulses*, Diploma thesis, Chemnitz University of Technology, Chemnitz, 2011, http://nbn-resolving.de/urn:nbn:de:bsz:ch1-qucosa-85485.
- R. Gorenflo, B. Hofmann, *On autoconvolution and regularization*, Inverse Problems 10 (1994), pp. 353–373.
- S. Koke, S. Birkholz, J. Bethge, C. Grebing, G. Steinmeyer, *Self-diffraction SPIDER*, Conference on Laser and Electro Optics (CLEO), San Jose, CA, 2008.

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Thank you for your attention! Are there any questions?