that of its neighbors. This classification arises naturally, the more so when we verify that it is finer than a classification based on C, $S(P)$ or the degrees, or any combination of these three criteria; it divides all tilings into 93 patterns (see [B, 1.7.7.8]).

1.H.4 Remark on Tile Markings

Here and elsewhere in the book we have used markings, or drawings, on the tiles, to reveal their orientation. It is reasonable to ask if we could have done without them, by considering only the forms of the tiles. It is often possible to draw the tiles with an irregular shape that bars certain symmetries. In fact, from the 93 tiling patterns mentioned above, only 12 cannot be represented by means of unmarked tiles. So if one is interested only in the shape of the tiles, there are only 81 patterns left.

Exercise 1. Find representations by means of unmarked tilings for the five crystallographic groups that are contained in $\mathrm{Is}^+(E)$ (see 1.G).

1.H.5

Finally, we remark that the connection between the notion of a crystallographic group and that of a tiling is not canonical. We have defined the group by means of the plane tiling it generates when we make it act on a well-defined previously chosen tile; in this process, there is no reason why all the vertices should be isometric, or even why they should all have the same structure. (We say that two vertices are *isometric* if there is an isometry taking a neighborhood of one onto a neighborhood of the other. In the figure below, S_1 and S_2 are not isometric.)

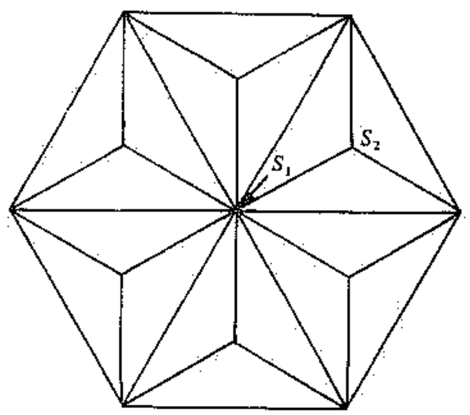


Figure 1.H.5.

We might equally well have the group act on a previously chosen *vertex*, thus obtaining a plane tiling that can contain different tiles. This kind of tiling is called *isogonal*, whereas the other kind is called *isohedral*. Using a classification based on an adjacency relation which is dual to that defined for isohedral tilings, we obtain again 93 types of isogonal tilings; of these, two are not representable by unmarked tilings (it is a bit more difficult, but still possible, to justify the marking of vertices in the same way as for tiles). Thus there are 91 (unmarked) isogonal tilings.

Note. Only tiles and vertices can give rise to classifications; the edges are always sided by the same number of tiles (namely, two), and bounded by the same number of vertices (again two).

Exercise 2. Find representations of the five crystallographic subgroups of $\mathrm{Is}^+(E)$ by means of unmarked isogonal tilings.

Problems

1.1 TRIANGLES AND QUADRILATERALS ([B, 1.9.14]). Does any triangle tile the plane? Any convex quadrilateral? Any quadrilateral?

1.2 FUNDAMENTAL DOMAIN OF A TILING ([B, 1.9.12]). Let $G \subset \mathrm{Is}(E)$ be a subgroup of the group of isometries of an affine Euclidean plane; we assume all orbits of G are discrete subgroups of E. For a fixed $a \in E$, show that G and the set P defined by

$$P = \{x \in E \colon \mathrm{d}(x,a) \leq \mathrm{d}(x, g(a)) \forall g \in G\}$$

verify the axioms of a crystallographic group.

1.3 THE FIVE ORIENTATION-PRESERVING CRYSTALLOGRAPHIC GROUPS ([B, 1.9.4]). For each of the five crystallographic groups containing only proper motions, find the following: the order of the stabilizers; the structure of the group; the different types of orbits; a presentation for the group (cf. [B, 1.8.7]).

1.4 THE ROBINSON NON-PERIODICAL TILING ([B, 1.9.16]). Show that with the six tiles below it is possible to tile the plane. Show also that any tiling with these tiles can never be periodic, i.e. its group of isometries cannot contain a non-trivial translation.

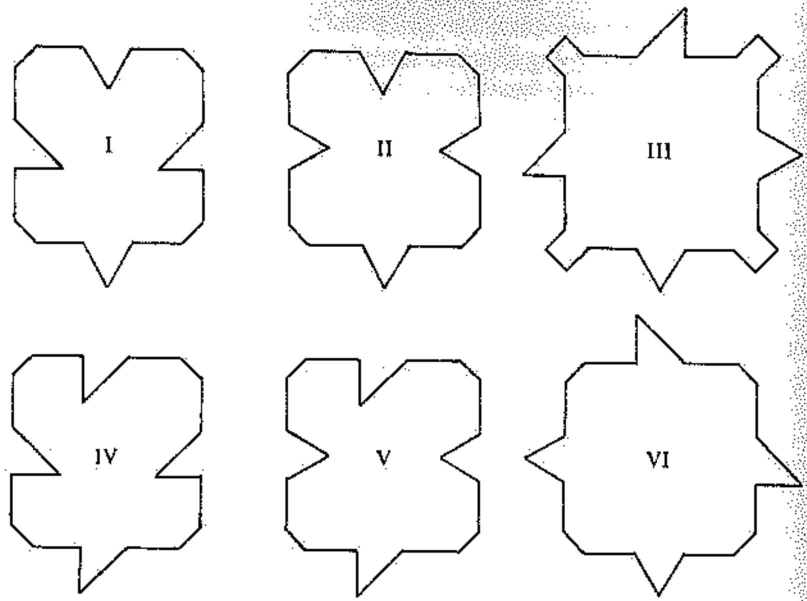


Figure 1.4

Chapter 2
Affine Spaces

All fields of scalars will be commutative.

2.A Affine Spaces; Affine Group ([B, 2.1, 2.3])

An affine space X is nothing more than a vector space under the action of the group generated by the linear automorphisms and the translations; this group is called the *affine group of* X and is *denoted* by $GA(X)$. Its elements are the (affine) automorphisms of X, or again the affine maps of X into itself. The elements of X are called *points*.

This amounts to eliminating the privileged role previously played by the origin of X, and making all points of X equal. There are of course more sophisticated definitions of the notion of affine spaces; see [B, 2.1.1, 2.1.6].

The other side of the coin is that now the elements of X are not vectors anymore, but merely points, so they cannot be added or multiplied by scalars any longer! The calculations which are still possible to perform in X (we shall distinguish the affine space X from the vector space that gave rise to it, which we will denote by $\vec{X}$) are the following: for $x, y \in X$ we can take the *midpoint* $(x+y)/2$ of x, y in X (if the field has characteristic different from 2); and for $x, y \in X$ we can find the vector $y - x = \overrightarrow{xy}$ in $\vec{X}$. And if x, y, z are on the same line and $x \neq y$ we can calculate the ratio $\overrightarrow{xz}/\overrightarrow{xy}$, which is a scalar.

As a result, we can also perform calculations of the type $x = a + \lambda\vec{u}$, where a and x are in X, the vector $\vec{u}$ is in $\vec{X}$, and λ is a scalar.

Last but not least, we can fix some $a \in X$ and consider X as a vector space

$x \in X$ are the scalars λ_i $(i=1,\ldots,n)$ such that

$$x = x_0 + \sum_i \lambda_i \overrightarrow{x_0 x_i} \quad \text{or} \quad \overrightarrow{x_0 x} = \sum_i \lambda_i \overrightarrow{x_0 x_i}.$$

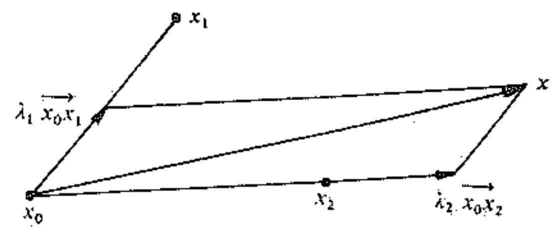


Figure 2. E.

An affine map is determined by its values on a simplex; in particular, for any two simplices in a finite-dimensional X, there is an automorphism taking one onto the other.

2.F The Fundamental Theorem of Affine Geometry ([B, 2.6])

This is the only delicate result following from the preceding notions. It concerns the set-theoretical bijections between two affine spaces X, X' (not taken *a priori* over the same field) that map lines into lines. Observe first that this condition is vacuous in dimension 1 and that, in the complex case, it is satisfied for maps which are not affine (for instance, $(z_1, z_2) \mapsto (\overline{z_1}, \overline{z_2})$). Our result says exactly that those two cases typify the only possible exceptions.

More precisely, if X and X' are defined over the fields K and K', a map f: $X \to X'$ is called *semi-affine* if the vectorialization $\vec{f}: X_a \to X'_{f(a)}$ of f is *semi-linear*, i.e. if there is field isomorphism σ: $K \to K'$ such that

$$\vec{f}(\lambda x + \mu y) = \sigma(\lambda)\vec{f}(x) + \sigma(\mu)\vec{f}(y)$$

for every $x, y \in X_a$ and every $\lambda, \mu \in K$.

The fundamental theorem of affine geometry says that if f: $X \to X'$ is a bijection that maps lines into lines, and if moreover X and X' have same finite dimension n with $n \geq 2$, then f is semi-affine.

2.G Finite-dimensional Real Affine Spaces ([B, 2.7])

From now on X will be a finite-dimensional affine space over the field of real numbers. These spaces (especially in dimension 1, 2 and 3) are those studied in classical geometry. They are especially rich and interesting as they relate to our

To begin with, X has a canonical topology, so we can talk about open sets, closed sets, compact sets and so on (see for instance problem 2.2). Differential calculus also applies in such spaces (see problem 2.4).

An important notion is that of a *half-space*; if Y is a hyperplane of X, its complement $X \setminus Y$ has exactly two connected components, which are called the *open half-spaces* determined by Y. Their closures are the corresponding *closed half-spaces*. This notion is fundamental in the study of convexity: see 11.B, 12.A.

Up to a positive scalar, a (finite-dimensional real) affine space possesses a canonical measure, called the *Lebesgue measure*. For example, if μ is such a measure, arising from the measure $\vec{\mu}$ on $\vec{X}$, we have the following definition for the *centroid* of a compact K in X with non-empty interior: If χ_K denotes the characteristic function of K and $\mu(K) = \int_X \chi_K \mu$ is the measure of K under μ, it can be shown that the vector integral

$$I_\mu(a) = \int_{x \in X} \chi_K \overrightarrow{ax}\, \vec{\mu}$$

is such that the point

$$a + (\mu(K))^{-1} I_\mu(a)$$

is independent of $a \in X$. This point is called the *centroid* of K; see problem 2.3.

A finite-dimensional vector space, and consequently a finite-dimensional affine space, can be *oriented*; this essentially amounts to choosing a basis and declaring it to be *positive*. Any other basis obtained from the first one by a linear automorphism of positive determinant is also positive. There are exactly two possible orientations, and neither is "canonical"; the choice is arbitrary.

Whether or not the real affine space X is oriented, we can always consider the maps $\vec{f} \in \mathrm{GL}(\vec{X})$ having positive determinant: det $\vec{f} > 0$. We say that they preserve orientation (whichever one we choose) and *denote* by $\mathrm{GA}^+(X)$ the subgroup of $\mathrm{GA}(X)$ formed by such maps. Its complement is *denoted* by $\mathrm{GA}^-(X)$, and contains all maps f such that det $\vec{f} < 0$.

Problems

2.1 THE THEOREMS OF CEVA AND MENELAUS ([B, 2.8.1, 2.8.2]). Let $\{a, b, c\}$ be a triangle in an affine plane, and let $a' \in \langle b, c\rangle$, $b' \in \langle c, a\rangle$, $c' \in \langle a, b\rangle$ be three points on the sides of this triangle. Prove that the three lines $\langle a, a'\rangle$, $\langle b, b'\rangle$, $\langle c, c'\rangle$ are concurrent (or parallel) if and only if we have

$$\frac{\overrightarrow{a'b}}{\overrightarrow{a'c}}\,\frac{\overrightarrow{b'c}}{\overrightarrow{b'a}}\,\frac{\overrightarrow{c'a}}{\overrightarrow{c'b}} = -1 \text{ (Theorem of Ceva)}$$

only if

$$\frac{\overrightarrow{a'b}}{\overrightarrow{a'c}}\frac{\overrightarrow{b'c}}{\overrightarrow{b'a}}\frac{\overrightarrow{c'a}}{\overrightarrow{c'b}}=1 \text{ (Theorem of Menelaus).}$$

2.2 CONNECTEDNESS OF COMPLEMENTS OF SUBSPACES ([B, 2.8.8]). Show that if X is a finite-dimensional real affine space and Y is a subspace the complement $X\setminus Y$ is connected if $\dim Y \le X-2$. Is $X\setminus Y$ simply connected? (See 18.A).

2.3 ASSOCIATIVE PROPERTIES OF CENTROIDS ([B, 2.8.11]). Let K be a compact subset of a finite-dimensional real affine space such that $\mathring{K}\neq\phi$; let H be a hyperplane of X and X', X'' its two closed half-spaces. We assume also that H is such that

$$\mathring{K}'\neq\phi \text{ and } \mathring{K}''\neq\phi,$$

where $K'=K\cap X'$ and $K''=K\cap X''$. Show (cf. 3.A) that cent(K) is the weighted average of cent(K') and cent(K'') with weights $\mu(K')$ and $\mu(K'')$, or, in the terminology of 3.A, the barycenter of the family $\{(\mathrm{cent}(K'),\mu(K')),(\mathrm{cent}(K''),\mu(K''))\}$. Deduce that if in addition K is convex, then $\mathrm{cent}(K)\in\mathring{K}$.

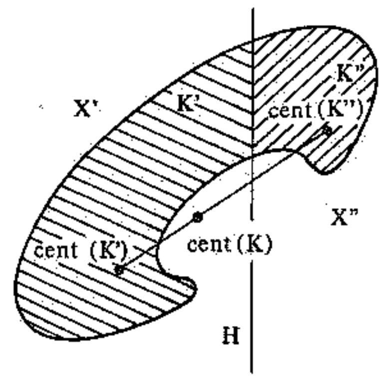


Figure 2.3.

2.4 EQUIAFFINE LENGTH AND CURVATURE ([B, 2.8.12]). Let X be a real affine plane, and *fix a basis for* $\vec{X}$. We can then define the determinant det $(\vec{u},\vec{v})$ of any two vectors in $\vec{X}$ relative to this basis by writing the 2×2 matrix whose columns are $\vec{u},\vec{v}$ in this basis. Let $c\colon [a,b]\to X$ be a differentiable curve of class C^3 in X. The *equiaffine length* of c (in X, relative to this basis) is the real number

$$\int_a^b\left(\det(\vec{c}'(t),\vec{c}''(t))\right)^{1/3}dt.$$

Show that c has same equiaffine length as $f\circ c$ for any $f\in \mathrm{SA}(X)$, where we put $\mathrm{SA}(X)=\{f\in \mathrm{GA}(X)\colon |\det\vec{f}|=1\}$ (see 2.B).

Show that we can reparametrize c by its equiaffine length if for every t we have $\det(\vec{c}'(t),\vec{c}''(t))\neq 0$. The *equiaffine curvature* is the number $K=\det(\vec{c}'',\vec{c}''')$ when c is parametrized by its equiaffine length; show that this curvature, too, is invariant by $\mathrm{SA}(X)$. Find the equiaffine length and curvature for an ellipse, a parabola or a hyperbola in X (always fixing a basis). In the same way that a curve in the Euclidean plane is determined up to an isometry by giving the curvature as a function of the arclength (see for example M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall 1976, p. 22, or M. Spivak, *Differential Geometry*, Publish or Perish 1970, vol. 2, p. 1-1), show that a curve in X is determined, up to an element of $\mathrm{SA}(X)$, by giving the equiaffine curvature as a function of its equiaffine arclength.

It follows classically that: First, the three medians of a triangle intersect and their intersection point is two thirds of the way from the vertices. Then the midpoints of the sides of a quadrilateral form a parallelogram ([B, 3.4.10]).

3.C Barycentric Coordinates ([B, 3.6])

Let $\{x_i\}_{i=0,1,\dots,n}$ be a simplex (cf. 2.E) in an affine space X of finite dimension n. Then every x in X can be uniquely written as $x=\Sigma_i\lambda_i x_i$ with $\Sigma_i\lambda_i=1$. The $n+1$ scalars which are thus uniquely determined are called the *barycentric coordinates of* x (in the simplex being considered).

In the case when $K=\mathbf{R}$, the set $\{\lambda x+(1-\lambda)y;\ \lambda\geq 0\}$ of barycenters of the pair (x, y) with positive mass is called the *segment* defined by x and y and is *denoted* by $[x, y]$. We will encounter this notion again, as well as the more general case of many positive masses, when we study convexity (11.A).

3.D A Universal Space ([B, 3.1, 3.2])

The preceding notions can be made more formal by the introduction of a *vector space* $\hat{X}$ attached to the affine space X; we define it as the union of points with a given non-zero mass (i.e. the product $X\times K^*$) and the vectors in $\vec{X}$ (cf. 2.A). In this space we can perform vector calculations; for instance, the elements of $\vec{X}$ correspond to the case $\Sigma_i\lambda_i=0$. The essential thing is that X is canonically embedded in $\hat{X}$ as the affine hyperplane formed by points with mass 1 (cf. 2.D). And the direction $\vec{X}$ of this hyperplane (in the sense of 2.D) is indeed the vector space which gives rise to X.

3.E Polynomials ([B, 3.3])

Starting from the classical notion of a homogeneous polynomial of degree k over a vector space, and using the vector space $\hat{X}$, we define for an affine space the notion of a *polynomial of degree less than or equal to* k over X. Conversely, it is possible to transform a polynomial of degree less than or equal to k over X into a polynomial of degree k over $\hat{X}$, by introducing a homogenizing variable (which is of course the mass). Moreover, a polynomial f over X possesses a *symbol* $\vec{f}$, which is a homogeneous polynomial over $\vec{X}$ and corresponds to the highest-degree term of f. In a less pedantic way, we can say that a polynomial of degree less than or equal to k is something which, in an arbitrary vectorialization of X, consists of a sum of homogeneous polynomials

Problems

3.1 ITERATED CENTERS OF MASS ([B, 3.7.16]). In a real affine space, consider p points $x_{1,1},\dots,x_{1,p}$ $(p\geq 2)$. For $i=1,2,\dots,p$ denote by $x_{2,i}$ the center of mass of the $(x_{1,j})_{j\neq i}$. Then define by recurrence the center of mass $x_{k+1,i}$ of the $(x_{k,j})_{j\neq i}$, for all $k\geq 1$. Prove that every sequence $(x_{k,i})_{k\in\mathbf{N}}$ converges. What can you say about the limit of these sequences for different values of i?

3.2 CENTROIDS OF THE BOUNDARY OF A TRIANGLE AND OF A FOUR-SIDED PLATE ([B, 3.7.13 and 3.7.14]). Determine the centroid of the physical object consisting of three homogeneous pieces of wire of same linear density, lying on the three sides of a triangle. Give a geometrical construction for this point.

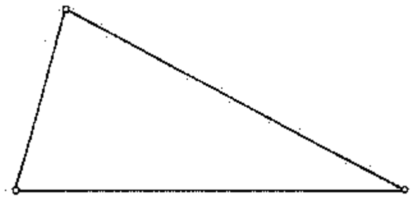


Figure 3.2:1.

Give a geometrical construction for the center of mass of a homogeneous plate in the shape of a quadrilateral. Compare this point with the centroid of the four vertices.

3.3 DIAMETERS OF THE BARYCENTRIC SUBDIVISIONS OF SIMPLICES ([B, 3.7.8]). Let Σ be a simplex in a Euclidean affine space of dimension n; its *diameter* d is the largest distance between any two of its points. Show that all simplices of the barycentric subdivision of Σ have diameter less than or equal to $nd/(n+1)$; deduce that when we iterate the process of barycentric subdivision, the diameter of all simplices tends towards zero.

3.4 DIRECT DEFINITION OF POLYNOMIAL MAPS ([B, 3.7.11]). Let X be an affine space and W a vector space. Find a direct definition (i.e. not using the universal space) for the space $\mathscr{P}_k(X;W)$ of polynomial maps of degree k from X into W. Show that your definition makes sense.

3.5 EXPLICIT CALCULATION OF THE POLAR FORM OF A POLYNOMIAL ([B, 3.7.15]). The polar form of a polynomial map f of degree k, from one vector space V into another vector space W, is the unique k-linear

Show that it is given by the formula

$$\varphi(v_1,\dots,v_k) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\sum_{1\le i_1\le\dots\le i_j\le k} f(v_{i_1}+\cdots+v_{i_j}). \qquad (*)$$

3.6 THE EULER IDENTITY ([B, 3.7.12]). Recall that a real-valued function is said to be of class C^n if it can be differentiated n times, and its n-th derivative is continuous. Let X be a real vector space and $f\colon X\to\mathbf{R}$ a C^1 map such that $f(\lambda x)=\lambda^k f(x)$ for all $x\in X$ and $\lambda\in\mathbf{R}$. Show that the derivative f' of f satisfies the *Euler identity*:

$$f'(x)(x)=kf(x)\quad \forall x\in X.$$

Write and prove an analogous form for the p-th derivative of f, when f is of class C^p and again homogeneous of degree k. Deduce that if f is of class C^k and homogeneous of degree k, it is necessarily a polynomial.

Chapter 4
Projective Spaces

All fields considered here are commutative.

Affine spaces have the serious inconvenience that certain theorems have exceptional cases, for instance when lines become parallel. Projective spaces were created by Desargues in 1639 to remedy this situation. The affine space is completed with points at *infinity* which correspond to the directions of straight lines. Then one needs to know how to come back to the original affine space. This program is carried out here in two steps: Chapters 4 and 5.

4.A Definition ([B, 4.1])

A *projective space* is the space of lines (i.e. one-dimensional vector subspaces) of a vector space. If E is the vector space, we will *denote* by $P(E)$ the associated projective space, formed by the lines of E. Algebraically, $P(E)$ is the quotient of the complement $E\setminus 0$ of the origin in E by the equivalence relation "$x\sim y$ if and only if there is a non-zero scalar k such that $y=kx$". The *dimension* of $P(E)$ is one less than the dimension of E.

EXAMPLES. We put $P^n(K)=P(K^{n+1})$ and call this space the n-dimen-

Problems

4.1 A MODEL FOR $P^3(\mathbf{Z}_2)$ ([B, 4.9.9]). Find a model, in the usual three-dimensional space, for the configuration formed by the points, lines and planes of $P^3(\mathbf{Z}_2)$. Draw pictures.

4.2 ORIENTABILITY OF REAL PROJECTIVE SPACES (first method) ([B, 4.9.4]). For a real projective space of finite dimension n, find the sign of the Jacobian of the transition maps

$$\pi_j \circ \pi_i^{-1} : (v_1, \ldots, v_n) \mapsto \left(\frac{v_1}{v_{j-1}}, \ldots, \frac{v_{i-1}}{v_{j-1}}, \frac{1}{v_{j-1}}, \frac{v_i}{v_{i-1}}, \ldots, \frac{v_{j-2}}{v_{j-1}}, \frac{v_j}{v_{j-1}}, \ldots, \frac{v_n}{v_{j-1}} \right)$$

(defined on the overlap $\mathbf{R}^n \setminus v_{j-1}^{-1}(0)$).

4.3 ORIENTABILITY OF REAL PROJECTIVE SPACES (second method) ([B, 4.9.5]). Find out whether $P^n(\mathbf{R})$ is orientable by studying the connectedness of the projective group $\mathrm{GP}(P^n(\mathbf{R}))$.

4.4 HYPERPLANES AND DUALITY ([B, 4.9.10]). Let $\{H_i\}$ be a family of hyperplanes in the projective space $P(E)$ of finite dimension n; find a relation between $\dim(\cap_i H_i)$ in $P(E)$ and $\dim(\langle \cup_i H_i \rangle)$ in $P(E^*)$.

4.5 NUMBER OF POINTS AND OF SUBSPACES IN A PROJECTIVE SPACE OVER A FINITE FIELD ([B, 4.9.11]). Let K be a field with k elements, and $P(E)$ a projective space of dimension n over K. Show that the cardinality of the set of p-dimensional subspaces of $P(E)$ is equal to

$$\frac{(k^{n+1}-1)(k^{n+1}-k)\cdots(k^{n+1}-k^p)}{(k^{p+1}-1)(k^{p+1}-k)\cdots(k^{p+1}-k^p)}.$$

Show that the order of the projective group $\mathrm{GP}(E)$ is

$$(k^{n+1}-1)(k^{n+1}-k)\cdots(K^{n+1}-k^{n-1})K^n.$$

4.6 MÖBIUS TETRAHEDRA ([B, 4.9.12 and 5.5.3]). Construct, in a three-dimensional projective space, two tetrahedra $\{a, b, c, d\}$ and $\{a', b', c', d'\}$ such that each vertex of the first belongs to a face of the second and vice versa (i.e. $a \in \langle b', c', d' \rangle$ etc. and $a' \in \langle b, c, d \rangle$ etc.)

Construct such tetrahedra in a very simple way by sending as many points to infinity as possible (cf. 5.D).

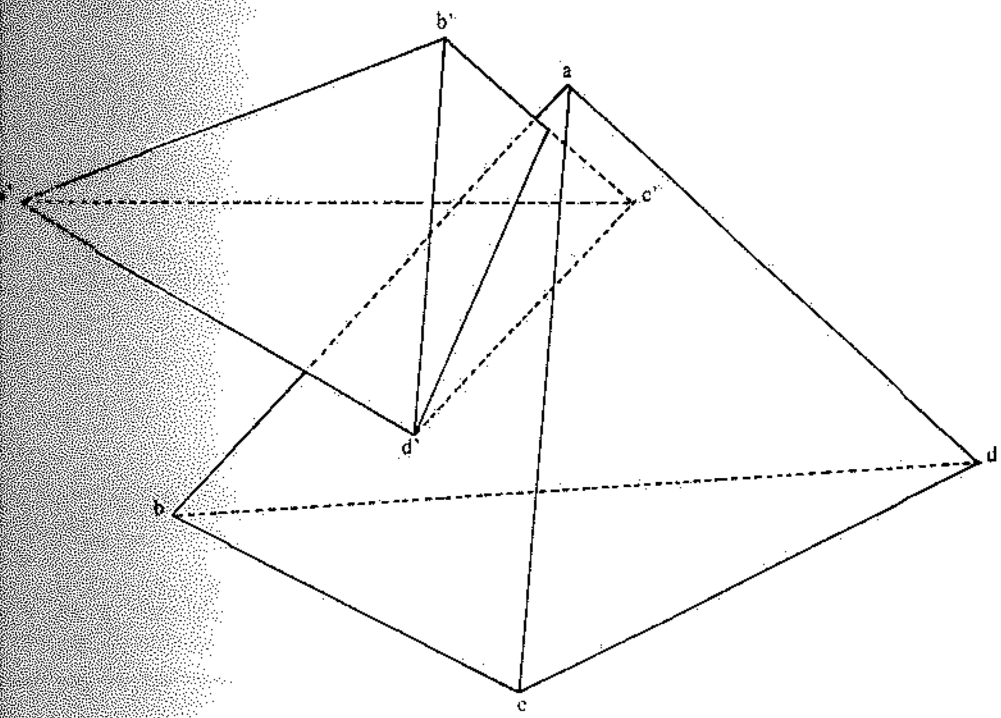


Figure 4.6.

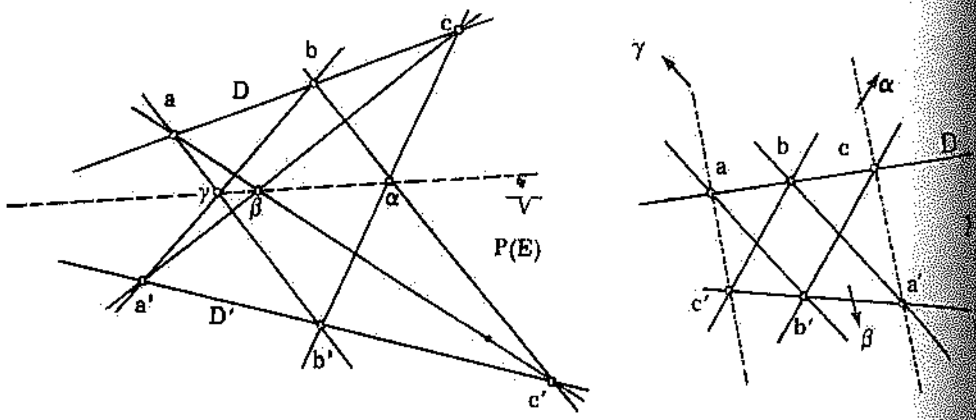


Figure 5.D.1.

THEOREM OF DESARGUES. *Let s, a, b, c, a', b', c' be points in an affine space such that s, a, a' (resp. s, b, b' and s, c, c') are aligned. Then if the three points $\langle a, b\rangle \cap \langle a', b'\rangle$, $\langle b, c\rangle \cap \langle b', c'\rangle$, $\langle c, a\rangle \cap \langle c', a'\rangle$ exist, they lie on the same line.*

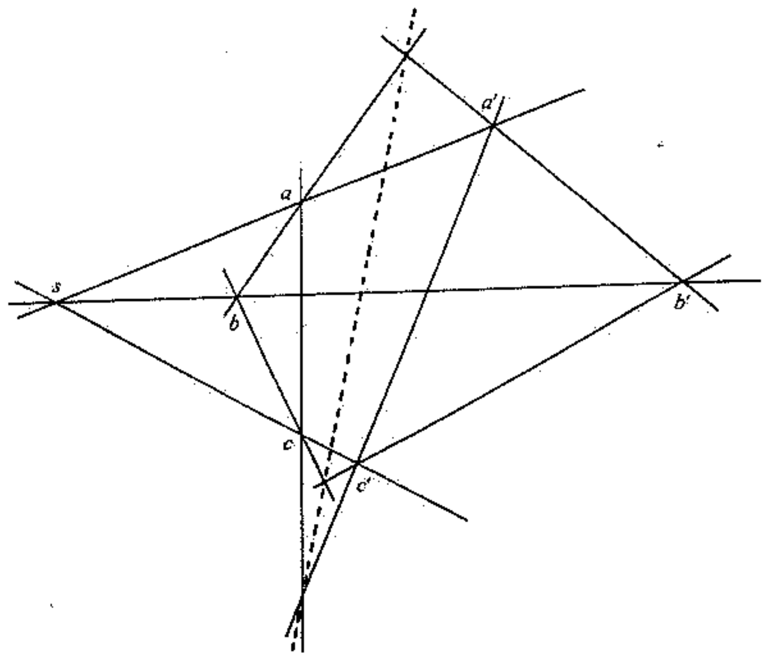


Figure 5.D.2.

Problems

5.1 THEOREM OF PAPPUS WHEN THERE ARE PARALLEL LINES ([B, 5.5.2]). Draw the figures for the theorem of Pappus (cf. 5.D) when there are points at infinity.

5.2 POINTS OUTSIDE YOUR DRAWING PAPER; RULER TOO SHORT ([B, 5.5.4 AND 5.5.5]). Suppose you are given a piece of paper with one point marked and segments of two lines which intersect outside the paper. Using only a straightedge (ruler), draw the line that joins the given point with the intersection point of the two lines.

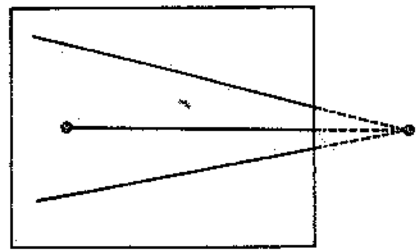


Figure 5.2.1.

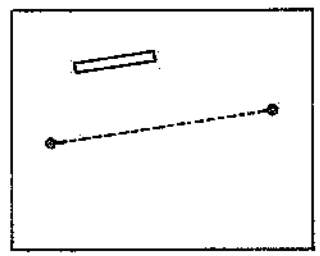


Figure 5.2.2.

Now suppose you are given two points, but your ruler is too short to connect them. Draw the line joining the points.

5.3 HEXAGONAL WEBS ([B, 5.5.8 AND 5.5.9]). We shall define a *web* in a real affine plane P as the following set of data: an open set A in P, and for each point a in A three distinct lines $d_i(a)(i=1,2,3)$ in P which go through a and which depend continuously on a. Show that for b on $d_1(a)$ close enough to a, we can define six points $(b_i)_{1,\ldots,6}$ as follows:

$$b_1 = d_3(b)\cap d_2(a), \quad b_2 = d_1(b_1)\cap d_3(a), \quad b_3 = d_2(b_2)\cap d_1(a),$$

$$b_4 = d_3(b_3)\cap d_2(a), \quad b_5 = d_1(b_4)\cap d_3(a), \quad b_6 = d_2(b_5)\cap d_1(a).$$

A web is said to be *hexagonal* if $b_6 = b$ for all sufficiently close a and b.

Let $(p_i)_{i=1,2,3}$ be three points of P, not on the same line, and let A be the complement of the three lines which connect each pair of points p_i. We define a web by setting $d_i(a) = \langle a, p_i\rangle$ for every $a \in A$. Show that this web is hexagonal.

More generally, consider a conic section C and a point p not situated on the conic. Assign to any point in the complement of $C \cup p$ the two tangents to C through x and the line xp. Show that the web thus obtained is hexagonal.

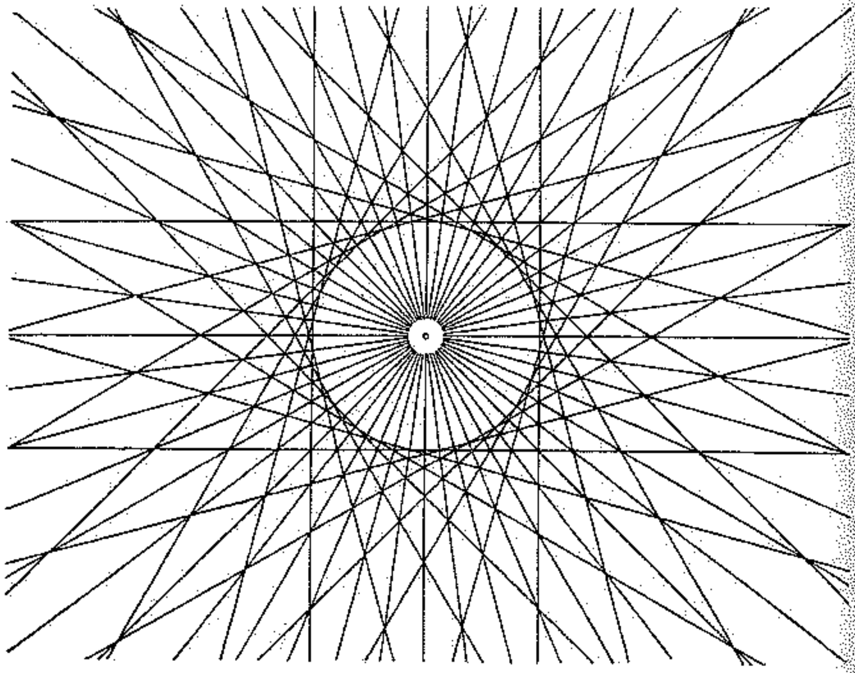


Figure 5.3.1.

Chapter 6

Projective Lines, Cross-Ratios, Homographies

6.A Cross-ratios ([B, 6.1, 6.2, 6.3])

To any four distinct points $(a_i)_{i=1,2,3,4}$ on a projective line (considered by itself or inside a projective space), we associate a scalar, *denoted* by $[a_i] = [a_1, a_2, a_3, a_4]$, and called the *cross-ratio* of these four points. For points on an affine line D, the cross-ratio is defined to be the same as on the completion $\tilde{D} = D \cup \infty_D$ (cf. 5.A).

The cross-ratio is a projective invariant, in the sense that if $(a_i)_{i=1,2,3,4}$ (resp. $(a_i')_{i=1,2,3,4}$) are four points on a line D (resp. D'), then the existence of a homography $f: D \to D'$ taking a_i to a_i', $i = 1,2,3,4$, is equivalent to $[a_i] = [a_i']$. In particular, cross-ratios are invariant under homographies.

There are several equivalent ways of defining cross-ratios. The first one is by putting $[a, b, c, d] = f(d)$, where $f(d)$ is the element of K given by the unique homography $f: D \to \tilde{K} = K \cup \infty$ (cf. 5.A and 4.E) from our line into $\tilde{K}$ satisfying $f(a) = \infty$, $f(b) = 0$ and $f(c) = 1$.

Second way: For $D = D' \cup \infty_D$, where D' is an affine line, put

$$[a, b, c, d] = \frac{\overrightarrow{ca}/\overrightarrow{cb}}{\overrightarrow{da}/\overrightarrow{db}}; \quad \text{in particular } [a, b, c, \infty_D] = \overrightarrow{ca}/\overrightarrow{cb}.$$

The following three relations describe the behavior of cross-ratios under permutations of the four points:

$$[a, b, c, d] = [b, a, c, d]^{-1} = [a, b, d, c]^{-1}, [a, b, c, d] + [a, c, b, d] = 1.$$

the convention being that

$$f(\infty)=\frac{\alpha}{\gamma} \quad \text{and} \quad f\left(-\frac{\delta}{\gamma}\right)=\infty$$

(unless $\gamma=0$, in which case $f(\infty)=\infty$).

If the field K is algebraically closed, f is studied by means of its fixed points. If there is only one of these, we put it at infinity; then f is a translation of the affine line obtained by taking infinity away. Otherwise, let a, b be the two distinct fixed points; then f satisfies $[a, b, m, f(m)]=$ constant for all m.

This constant value is λ/μ, where λ and μ are the two eigenvalues of the matrix of f, namely

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

An *involution* of a projective line is by definition a homography different from the identity, whose square is the identity. The analytical condition for a homography is

$$\operatorname{trace}(A)=\alpha+\delta=0.$$

An involution is determined by its value on two points. Every homography is the product of at most three involutions. If K is algebraically closed and a, b are the two fixed points of an involution, then $[a, b, m, f(m)]=-1$ for every m (cf. 3.B).

Problems

6.1 RELATION BETWEEN THE CROSS-RATIOS OF FIVE POINTS ([B, 6.8.1]). Let x, y, z, u, v be five points on the same projective line. Show that the following always holds:

$$[x, y, u, v][y, z, u, v][z, x, u, v]=1.$$

6.2 RICATTI DIFFERENTIAL EQUATIONS ([B, 6.8.12]). If $a, b, c:[\alpha,\beta] \to \mathbf{R}$ are continuous functions, we consider the differential equation (called a Ricatti equation): $y'(t)=a(t)y^2+b(t)y+c(t)$; show that if $y_i(i=1,2,3,4)$ are four solutions of this equation, the cross-ratio $[y_i(t)]$ is independent of t.

6.3 DUALITY IN A TETRAHEDRON ([B, 6.8.21]). Let T be a tetrahedron in a three-dimensional projective space and let D be a line. Show that the cross-ratio of the four intersection points of D with the faces of T is equal to the cross-ratio of the four planes passing through D and the vertices of T.

6.4 FIVE-POINT SETS ON A PROJECTIVE PLANE ([B, 6.8.17]). Let $(a_i)_{i=1,\ldots,5}$ be points on a projective plane, so that the first four form a projective base. We *denote* by d_{ij} the line $\langle a_i, a_j\rangle$. Show that the following holds:

$$[d_{12}, d_{13}, d_{14}, d_{15}][d_{23}, d_{21}, d_{24}, d_{25}][d_{31}, d_{32}, d_{34}, d_{35}]=1.$$

Show that a necessary and sufficient condition for the existence of a homography taking the $(a_i)_{i=1,\ldots,5}$ into new points $(a_i')_{i=1,\ldots,5}$ is that the following two equalities be satisfied:

$$[d_{12}, d_{13}, d_{14}, d_{15}]=[d_{12}', d_{13}', d_{14}', d_{15}']$$

and

$$[d_{23}, d_{21}, d_{24}, d_{25}]=[d_{23}', d_{21}', d_{24}', d_{25}'],$$

where we have put $d_{ij}'=\langle a_i', a_j'\rangle$. Generalize for the case of a projective space of arbitrary dimension.

6.5 EIGENVALUES OF A HOMOGRAPHY ([B, 6.8.7]). Let f be a homography with two distinct fixed points a, b; show that the pair $\{k, 1/k\}$, where $k=[a, b, m, f(m)]$ for every m, depends only on f and not on the choice of the order of a, b. If f has matrix $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ show that $\{k, 1/k\}$ are the roots of the equation

$$(\alpha\delta-\beta\gamma)k^2-(\alpha^2+2\beta\gamma+\delta^2)k+(\alpha\delta-\beta\gamma)=0.$$

6.6 CLASSIFICATION OF COMPLEX HOMOGRAPHIES ([B, 6.6.8]). The data are those of 6.5, and moreover $K=\mathbf{C}$. We say that f is *elliptic* if the complex number k (or $1/k$) has absolute value 1, *hyperbolic* if k is positive real, and *loxodromic* otherwise. Show that, normalizing $M(f)$ by $\alpha\delta-\beta\gamma=1$, we can characterize these three cases by using the trace t of f, given by $t=\alpha+\delta$:

$$\begin{aligned}
f \text{ elliptic} &\iff t \text{ is real and } |t|<2;\\
f \text{ hyperbolic} &\iff t \text{ is real and } |t|>2;\\
f \text{ loxodromic} &\iff t \text{ is not real.}
\end{aligned}$$

Study, for the three cases considered, the nature of the iterates $f^n (n\in\mathbf{Z})$.

8.J Orientation, Vector Products, Gram Determinants ([B, 8.11])

In this section we consider an *oriented* n-dimensional Euclidean vector space. Such a space possesses a *canonical volume form* λ_E, defined by the conditions that it is an exterior n-form (i.e. an alternating multilinear n-form), and its value $\lambda_E(e_1,\dots,e_n)$ on any positive orthonormal basis is 1. In dimension 3 this form is often called the *mixed product* and is *denoted* simply by $\lambda_E(x,y,z)=(x,y,z)$.

The *vector product* of $n-1$ vectors $x_1,\dots,x_{n-1}$ in E is the vector $x_1\times\cdots\times x_{n-1}$ in E defined by the following duality relation:

$$(x_1\times\cdots\times x_{n-1}|y)=\lambda_E(x_1,\dots,x_{n-1},y)\quad \text{for all } y\in E.$$

The vector product is zero if and only if the x_i are linearly dependent. Otherwise, it is orthogonal to all the x_i, and added to them it forms a positive basis; finally, its norm is given by

$$\|x_1\times\cdots\times x_{n-1}\|^2=\mathrm{Gram}(x_1,\dots,x_{n-1}),$$

where $\mathrm{Gram}(x_1,\dots,x_p)$ *denotes*, in general, the determinant

$$\mathrm{Gram}(x_1,\dots,x_p)=\det\big((x_i|x_j)\big)=\begin{vmatrix}\|x_1\|^2 & (x_1|x_2) & \cdots & (x_1|x_p)\\ (x_2|x_1) & \|x_2\|^2 & \cdots & (x_2|x_p)\\ \vdots & \vdots & & \vdots\\ (x_p|x_1) & (x_p|x_2) & \cdots & \|x_p\|^2\end{vmatrix}$$

Finally, observe the following relation, very useful in calculating volumes: $\mathrm{Gram}(x_1,\dots,x_n)=(\lambda_E(x_1,\dots,x_n))^2$.

Problems

8.1 AN IRREDUCIBLE GROUP CAN LEAVE INVARIANT AT MOST ONE EUCLIDEAN STRUCTURE ([B, 8.12.1]). Let E be a finite-dimensional real vector space, φ and ψ two Euclidean structures on E, and $G\subset \mathrm{GL}(E)$ a subgroup of the linear group of E; we assume G is irreducible (cf. [B, 8.12.2]). Show that if $G\subset O(E,\varphi)\cap O(E,\psi)$ (i.e. if every element of G leaves φ and ψ invariant, cf. 13.E), then φ and ψ are proportional.

8.2 A VECTOR ISOMETRY ALWAYS POSSESSES AN INVARIANT LINE OR PLANE ([B, 8.12.2]). To prove that $f\in O(E)$ always leaves some line or plane invariant, consider some $x\in S(E)$ such that $\|f(x)-x\|$ is minimal, and show that $x, f(x), f^2(x)$ lie in the same plane.

8.3 DIVIDING AN ANGLE BY n ([B, 8.12.7]). Show that, for every $n\in\mathbf{N}^*$ and every $a\in\tilde{\mathrm{U}}(E)$, the equation $nx=a$ has exactly n solutions in $\tilde{\mathrm{U}}(E)$. Draw the solutions on a circle for a few values of a, with $n=2,3,4,5$.

8.4 FIND THREE LINES WHOSE BISECTORS ARE GIVEN ([B, 8.12.20]). We call a *bisector* of two lines A, B in a Euclidean vector space any bisector of A and B in the plane generated by the two lines (cf. 8.F).

Let S,T,U be three lines in a 3-dimensional Euclidean vector space. Find three lines A,B,C such that line S is a bisector of A,B, line T is a bisector of B,C, and line U is a bisector of C,A. Study the possible generalizations of this problem: replacing lines by half-lines, considering more than three lines, or considering higher-dimensional spaces.

8.5 AUTOMORPHISMS OF $\mathbf{H}$ ([B, 8.12.11]). Show that every automorphism of $\mathbf{H}$ is of the form $a\mapsto \mathscr{R}(a)+\rho(\mathscr{P}(a))$, where $\rho\in O^+(3)$.

8.6 VECTOR PRODUCTS IN R^3 ([B, 8.12.9]). For every $a,b,c\in\mathbf{R}^3$, prove the following formulas:

$$a\times(b\times c)=(a|c)b-(a|b)c, \tag{1}$$

$$(a\times b, a\times c, b\times c)=(a,b,c)^2, \tag{2}$$

$$(a\times b)\times(a\times c)=(a,b,c)a. \tag{3}$$

Show that $\mathbf{R}^3$, endowed with the operations of addition and vector product, is an anticommutative algebra which, instead of being associative, satisfies the *Jacobi identity*

$$a\times(b\times c)+b\times(c\times a)+c\times(a\times b)=0.$$

Such an algebra is called a *Lie algebra*.

If p,q,r denote the projections from $\mathbf{R}^3$ onto the three coordinate planes, show that

$$\|a\times b\|^2=\mathrm{Gram}(p(a),p(b))+\mathrm{Gram}(q(a),q(b))+\mathrm{Gram}(r(a),r(b)).$$

Find a geometrical interpretation for this result (see the definition of the Gram determinant in 8.J).

Study the equation $x \times a = b$ (for a and b given); find whether there is a solution and whether it is unique.

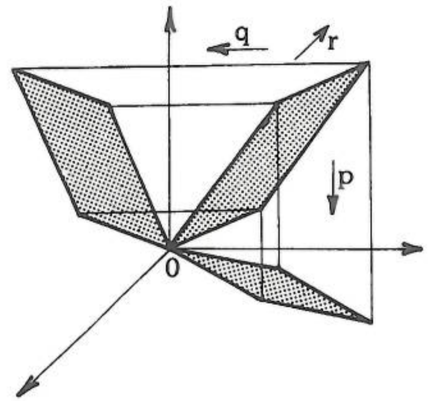


Figure 8.6.

Chapter 9

Euclidean Affine Spaces

9.A Definitions ([B, 9.1])

We consider a real affine space X of finite dimension (which is always den by n), and whose underlying vector subspace $\vec{X}$ (see 2.A) is endowed wi Euclidean structure; we say that X is a *Euclidean affine space*. The stan example is $\mathbf{R}^n$, considered as an affine space.

We make X into a metric space by taking the *distance function* $d(x, y) =$ which is generally written simply $d(x, y) = xy$. The triangle inequality is si which means that $xz = xy + yz$ implies that y belongs to the segment [(cf. 3.C). A Euclidean affine space possesses a canonical topological struc (see 2.G), whose compact sets are the closed sets bounded in the metric d

The group of *isometries* of X, i.e. the bijections of X such that $f(x)f(y)$ for every x, y in X, is *denoted* by $\mathrm{Is}(X)$. We have the following fundam fact: an isometry is necessarily an affine map, or, more precisely,

$$\mathrm{Is}(X) = \{ f \in \mathrm{GA}(X) \text{ and } \vec{f} \in O(X) \} \text{ (cf. 2.B)}.$$

We can then define (see 2.G) $\mathrm{Is}^{\pm}(X) = \mathrm{Is}(X) \cap \mathrm{GA}^{\pm}(X)$; the element $\mathrm{Is}^{+}(X)$ are called (proper) motions, and those of $\mathrm{Is}^{-}(X)$ are sometimes c improper motions.

9.B Subspaces ([B, 9.2])

Then we can talk about the volume $\mathscr{L}(K)$ of a compact K in X, defined as

$$\mathscr{L}(K)=\int_X \chi_K m\mu,$$

where χ_K is the characteristic function of K.

The reader can find in [**B**, 9.12; 9.13] some important results about volumes, the Stein symmetrization and the isodiametric inequality.

Problems

9.1 SYLVESTER'S THEOREM ([B, 9.14.25]). Prove that if a set of n points $(x_i)_{i=1,\dots,n}$ on an affine plane has the property that every straight line containing two of the x_i also contains a third, then all the points are on the same line.

9.2 BISECTORS ([B, 9.14.3]). Let D, D' be two non-parallel lines in a Euclidean plane X; show that $\{x \in X : d(x, D) = d(x, D')\}$ is formed by the two bisectors of D, D'. What is this set when X is higher-dimensional?

9.3 LIGHT POLYGONS ([B, 9.14.33]). Let C be a plane curve in a Euclidean plane, of class C^1 (cf. 3.6) and strictly convex. Show that for any integer $n \geq 3$ there is at least one n-sided *light polygon* inscribed in C, i.e. a polygon for each vertex of which the exterior bisector of the two sides meeting at this vertex coincides with the tangent to C at this point.

9.4 FINDING THE CENTER OF A SIMILARITY ([B, 9.14.40]). Given four points a, b, a', b' in $\mathbf{R}^2$, construct the centers of the similarities taking a to a' and b to b'.

9.5 INSCRIBING A SQUARE IN A TRIANGLE ([B, 9.14.16]). Given a triangle, inscribe a square inside it: see figure below.

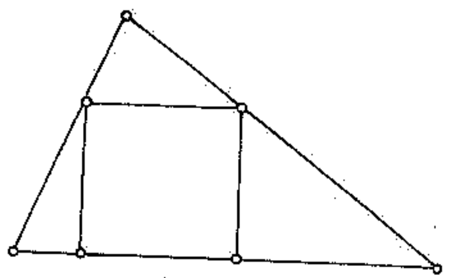


Figure 9.5.1.

9.6 CONVEXITY OF PASCAL LIMAÇONS ([B, 9.14.18]). Given a circle and a point, we call a *Pascal limaçon* the curve obtained by projecting this point to all the tangents to the circle (in general, a curve obtained from another by this procedure is called its "pedal curve"). Study the convexity characteristics of Pascal limaçons as a function of the position of the point relative to the circle.

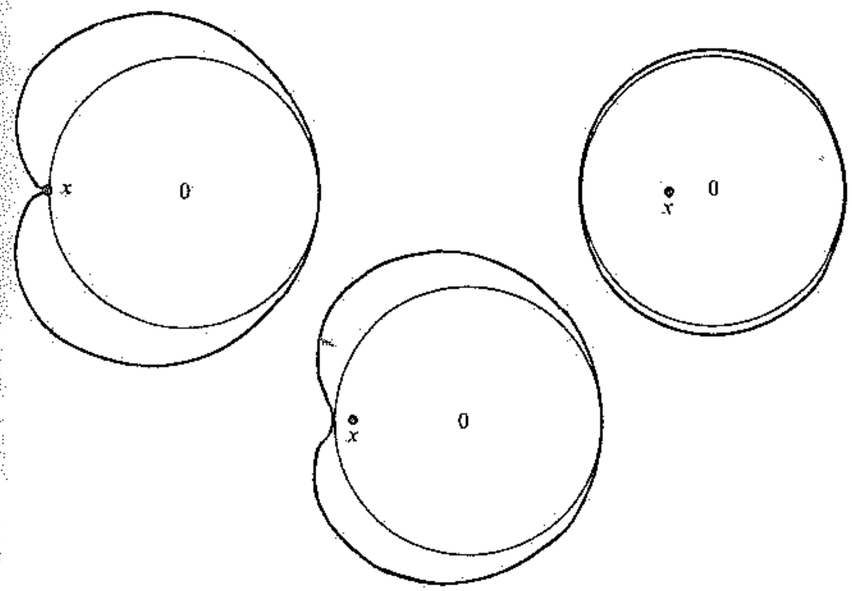


Figure 9.6.1.

Chapter 11

Convex Sets

We'll be working in a real affine space X of real dimension d.

11.A Definition; First Properties ([B, 11.1, 11.2])

A subset S of X is called *convex* if the segment $[x, y]$ (cf. 3.C) is contained in S for every $x, y \in S$. A weaker condition is that defining star-shaped sets: S is *star-shaped with center* $a \in X$ if $[a, x] \subset S$ for every $x \in S$. In **R** convex sets are the same as intervals.

An arbitrary intersection of convex sets is convex; in particular every subset A of X gives rise to a smallest convex set containing it, which we'll *denote* by $\mathscr{E}(A)$ and call *convex hull* of A. The convex hull $\mathscr{E}(A)$ of A is characterized as the set of barycenters of families $\{(x_i, \lambda_i)\}$, where each x_i is in A and each $\lambda_i \geq 0$. A theorem of Caratheodory asserts that this holds even if we take only families of at most $d+1$ elements.

If S is convex, so are its closure $\bar{S}$ and its interior $\mathring{S}$. The *dimension* of a convex set is that of the smallest affine subspace $\langle S \rangle$ containing it; saying that $\dim S = d$ is the same as saying that S has non-empty interior.

11.B The Hahn-Banach Theorem. Supporting Hyperplanes ([B, 11.4, 11.5])

The Hahn-Banach theorem states the following: Let A be a non-empty open convex set in X, and let L be an affine subspace of X such that $A \cap L \neq \emptyset$. Then there is a hyperplane of X which contains L and does not meet A.

We immediately conclude that through each point in the boundary of a closed convex set S there passes at least one *supporting hyperplane* of S, i.e. a hyperplane H intersecting the boundary of S and such that S is entirely contained in one of the closed subspaces defined by H (cf. 2.G).

Next, we can study the following polarity operation for the convex sets (compare with 10.**B** and 14.E): We assume X is a Euclidean vector space with origin O. We associate to every subset A of X its *polar reciprocal* A^*, defined by

$$A^* = \{\, y \in X : (x|y) \leq 1 \quad \text{for every } x \in A \,\}.$$

The correspondence $A \mapsto A^*$ is a good duality whenever A is a compact convex set containing O in its interior; we have $(A^*)^* = A$ and the support hyperplanes of A are the polar hyperplanes of the points of A^*, relative to the unit sphere $S(0,1)$.

Finally, the Hahn-Banach theorem implies Helly's theorem: For X a (d-dimensional) affine space, let $\mathscr{F}$ be a family of convex compact subsets of X such that the intersection of any $d+1$ elements of $\mathscr{F}$ is non-empty; then the intersection of all the elements of $\mathscr{F}$ is non-empty as well.

11.C Boundary Points of a Convex Set ([B, 11.6])

They can be classified in many ways; one of the most useful is the following: A point x in the boundary of a convex set A is called *extremal* if whenever $x = (y+z)/2$ with $y, z \in A$ we have $y = z$. The theorem of Krein and Milman says that a convex compact set is the convex hull of its extremal points.

Problems

11.1 PARTITION OF THE PLANE INTO CONVEX SETS ([B, 11.9.22]). Find all partitions of the plane into two convex sets.

11.2 EXTREMAL POINTS IN TWO DIMENSIONS ([B, 11.9.8]). Prove that the extremal points of a convex set in the plane form a closed set.

11.3 HILBERT GEOMETRY ([B, 11.9.8]). Let A be a convex compact set of X whose interior is non-empty. Given two distinct points x, y of $\mathring{A}$, put $d(x, y) = |\log[x, y, u, v]|$, where u, v are the two points where the line $\langle x, y \rangle$ meets the boundary of A, and $[\cdot,\cdot,\cdot,\cdot]$ denotes the cross-ratio (cf. 6.A or [B, 6]). Show that $d : \mathring{A} \times \mathring{A} \to \mathbf{R}$, defined by the equation above and $d(x, x) = 0$, $\forall x \in \mathring{A}$, is a metric. Show that this metric is excellent (9.G). Study the relation between the strict triangle inequality (9.A) and the nature of the boundary

11.4 THE LUCAS THEOREM ([B, 11.9.21]). Let P be a polynomial with complex coefficients, and let P' be its derivative. Show that in the affine space $\mathbf{C}$, all the roots of P' belong to the convex hull of the roots of P. When P has degree three and distinct roots a, b, c, show that there is an ellipse inscribed in the triangle $\{a, b, c\}$ and whose foci are the roots u, v of P'.

11.5 STAR-SHAPED SETS ([B, 11.9.20]). Given a subset A of an affine space X, consider the set $N(A)$ of points a such that A is star-shaped with center a. Show that $N(A)$ is convex. Find $N(A)$ for a number of shapes of A.

Chapter 12

Polytopes; Compact Convex Sets

12.A Polytopes ([B, 12.1, 12.2, 12.3])

We'll be working in a d-dimensional real affine space X, for d finite. A *polytope* is a convex compact set with non-empty interior, which can be realized as the intersection of a *finite* number of closed half-spaces of X (cf. 2.G). We shall assume there are no *superfluous* half-spaces in the intersection. For $d = 2$ we use the word *polygon*.

The *faces* of a polytope P are the intersections of its boundary with the hyperplanes that define P. A face is itself a polytope (of dimension $d-1$) inside the hyperplane which contains it. We define by induction the k-faces of P as the faces of all the $(k+1)$-faces of P; 1-faces are called *edges* and 0-faces are called *vertices*.

$$\boxed{\text{From now on } X \text{ is Euclidean affine.}}$$

Give a polytope P, we can define its *volume* $\mathcal{L}(P)$ and its *area* $\mathfrak{A}(P)$; the area is the sum of the volumes of all the faces (considered as $(d-1)$-polytopes).

12.B Convex Compact Sets ([B, 12.9, 12.10, 12.11])

A convex compact set whose interior is non-empty can be approximated, in the two notions possess very nice properties. For example, for every real positive λ we put

$$B(C,\lambda) = \{x \in X : d(x,C) \le \lambda\}.$$

Then

$$\mathfrak{A}(C) = \lim_{\lambda \to 0} \frac{\mathscr{L}(B(C,\lambda)) - \mathscr{L}(C)}{\lambda}$$

Next, the isoperimetric inequality:

$$\frac{\mathfrak{A}(C)}{\alpha(d)} \ge \left(\frac{\mathscr{L}(C)}{\beta(d)}\right)^{(d-1)/d},$$

which holds for every convex compact set with non-empty interior; here $\alpha(d)$ (resp. $\beta(d)$) denote the area (resp. the volume) of the unit sphere in $\mathbb{R}^d$. Moreover, equality only holds if C is a sphere.

Notice that the two above results can be generalized for subsets of X which are "nice" enough though not convex: for example differentiable manifolds (cf. problem 12.3).

12.C Regular Polytopes ([B, 12.4, 12.5, 12.6])

A polygon is called *regular* if all its sides have same length and all its angles are equal. For $n \ge 3$ there are always n-sided regular polygons, and all n-sided regular polygons are similar. A regular polygon can always be inscribed in a circle. Its symmetry group possesses $2n$ elements, and is called the *dihedral group* of order $2n$; it acts simply transitively (cf. 1.D) on the pairs formed by a vertex and a side that ends at this edge. See problem 12.1.

The easiest way to generalize this notion for $d \ge 3$ is the following: consider the d-tuples $(F_0, F_1, \dots, F_{d-1})$ such that the F_i are i-faces and $F_{i-1} \subset F_i$, $1 \le i \le d-1$. Then the polytope P is *regular* if its isometry group $G(P)$ acts transitively on the set of such d-tuples, in which case the action is also *simply transitive* (cf. 1.D). A regular polytope can always be inscribed in a sphere.

The following are examples of regular polytopes: the *cocube* Coc_d whose vertices are $\pm e_i$ $(i = 1, \dots, d)$ in an orthonormal basis of X; it has $2d$ vertices and 2^d faces. The *cube* Cub_d has as vertices the 2^d points whose coordinates are given by $(\pm 1, \dots, \pm 1)$ (in an orthonormal basis), and it has $2d$ faces. And the *regular simplex* Sim_d which, considered in the hyperplane $\sum_{i=1}^{d+1} x_i = 1$, has as vertices the $d+1$ points e_i of an orthonormal basis; it also has $d+1$ faces.

It can be shown that for $d \ge 5$ these are the only regular polytopes (up to a similarity, of course). On the other hand, for $d = 3$ there are two exceptional regular polytopes, the dodecahedron and the icosahedron (Figure 1.F), and for $d = 4$ there are three exceptions. The existence of these exceptional polytopes is not obvious; the problem can be reduced to the case $d = 3$. Problem 19.1 gives

a construction of the regular icosahedron; another one consists in cleverly placing regular pentagons on a cube ([B, 12.5.5]), and finally one can leave aside all subtlety and just give the coordinates for the vertices: $(0, \pm\tau, \pm 1), (\pm 1, 0, \pm\tau), (\pm\tau, \pm 1, 0)$, where $\tau = (\sqrt{5}+1)/2$.

Problems

12.1 REGULAR PENTAGON ([B, 12.12.4]). Justify the following two constructions for the regular pentagon:

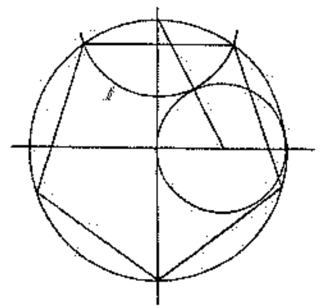


Figure 12.1.1.

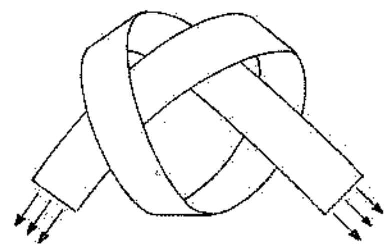


Figure 12.1.2.

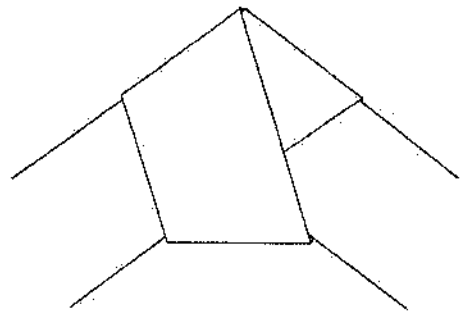


Figure 12.1.3.

12.2 ORDER OF THE GROUP OF A REGULAR POLYHEDRON ([B, 12.12.10]). Show that, for a regular polyhedron (i.e. a 3-dimensional polytope), the order of its group of isometries is equal to four times the number of its edges.

12.3 THEOREMS OF GULDIN ([B, 12.12.10.9]). Consider a compact set K of a plane P in the 3-dimensional Euclidean space E. Show that the volume of the compact set of C of E, generated by rotating K around a line D of P which does not intersect K, is given by the formula

$$\mathscr{L}_E(C) = 2\pi \cdot d(g, D) \cdot \mathscr{L}_P(K),$$

where $g = \operatorname{cent}(K)$ denotes the centroid of K (see 2.G).

If the boundary of K is considered as a homogeneous wire and h is the center of mass of this wire (in the usual sense), show that the area of C is given by the formula

$$\mathfrak{A}_E(C) = 2\pi \cdot d(h, D) \cdot \mathfrak{A}_P(K).$$

(Both areas are understood in the sense of differentiable manifolds.)

Find applications of this formula, as well as special cases of volumes or areas already known.

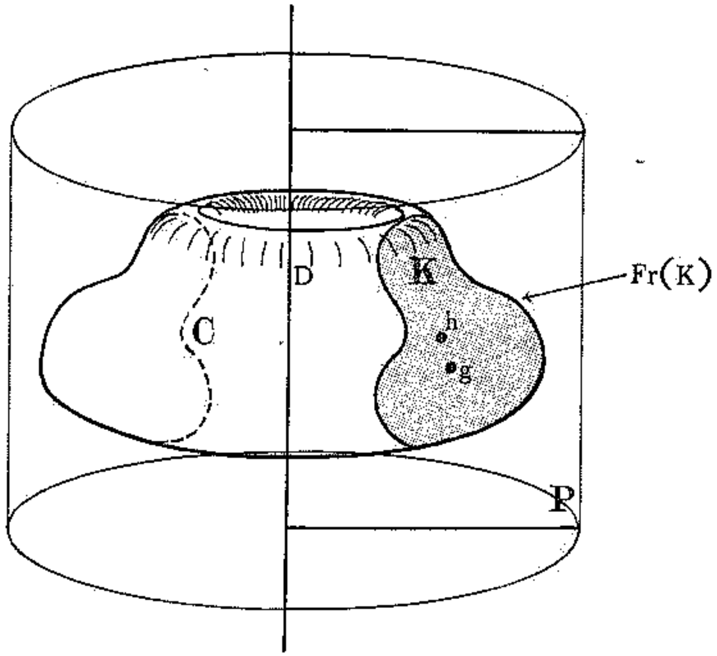


Figure 12.3.1.

12.4 VOLUME OF POLAR RECIPROCALS OF ELLIPSOIDS ([B, 12.12.2]). Show that if E is an ellipsoid in the Euclidean vector space X, containing O in its interior, then its polar reciprocal E^* (see 11.B) is an ellipsoid with the same property. Their volumes satisfy $\mathscr{L}(E)\mathscr{L}(E^*) \geq (\beta(d))^2$, and equality takes place if and only if O is the center of E (for the definition of $\beta(d)$, see 12.B or [B. 9.12.4]).

12.5 THE BLASCHKE ROLLING THEOREM ([B, 12.12.4]). Let C be a compact convex set in the plane whose boundary is a biregular curve (cf. M. Berger and B. Gostiaux, *Géométrie Différentielle*, Armand Colin, 1972, p. 309) of class C^2. Let A (resp. a) be a point on the boundary of C where the curvature is maximal (resp. minimal). Show that the osculating circle γ at a can roll all around the boundary, always staying inside C, and the boundary can roll all around the osculating circle Γ at A. Is this still true if we replace γ by the largest circle contained in C or Γ by the smallest circle containing C?

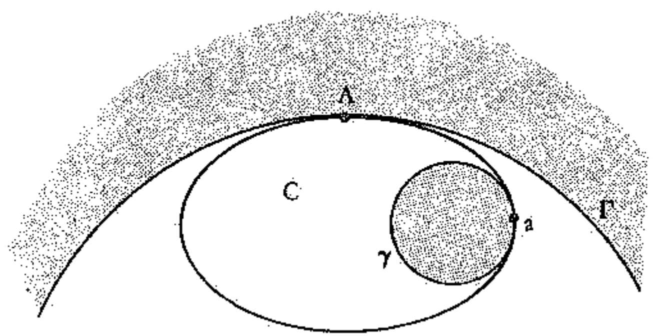


Figure 12.5.1.

of E containing the elements y such that $P(x,y)=0$ for every $x \in E$. Thus we have $\operatorname{rank}(q)+\dim(\operatorname{rad}(q)) = \dim E$. We say that q is *non-degenerate* if its rank is equal to n, and *degenerate* otherwise.

An important difference between the general and the Euclidean case is that here the restriction of a non-degenerate form q to a subspace can be degenerate. A subspace F of E is called *singular* (resp. *non-singular*) if $q|_F$ is degenerate (resp. non-degenerate); it is called a *null subspace* if $q|_F = 0$, i.e. if q is identically zero over F.

The *isotropic cone* of q is the subset $q^{-1}(0)$ of E; its elements are called *isotropic* vectors. An *anisotropic* form is one for which the isotropic cone consists of the zero vector only.

$$\boxed{\text{From now on } q \text{ will be assumed non-degenerate.}}$$

13.D Orthogonality ([B, 13.3])

Let E be a vector space endowed with a quadratic form, and F a subspace; the *orthogonal complement* of F, *denoted* by $F^\perp$, is the subspace of E defined by $F^\perp = \{P(x,y)=0 \text{ for all } x \in F\}$. In general, unlike the Euclidean case, the sum $F+F^\perp$ is not always direct; but the following properties hold for all subspaces F, F':

$$(F^\perp)^\perp = F; \quad \dim F + \dim F^\perp = \dim E;$$

$$\operatorname{rad} F = F^\perp \cap F; \quad F \text{ is a null subspace} \Leftrightarrow F \subset F^\perp,$$

$$(F \cap F')^\perp = F^\perp + F'^\perp, \quad (F+F')^\perp = F^\perp \cap F'^\perp.$$

$$E = F \oplus F^\perp \text{ (direct sum)} \Leftrightarrow F \text{ is non-singular}$$

$$\Leftrightarrow F^\perp \text{ is non-singular} \Leftrightarrow F \cap F^\perp = 0.$$

13.E The Group of a Quadratic Form ([B, 13.6, 13.7])

The *orthogonal group* of (E,q), *written* $O(E)$ or $O(q)$, is defined as

$$O(E) = \{ f \in \mathrm{GL}(E) : f^* q = q \}.$$

For a basis in which the matrix of q is A, the matrices S of the elements f of $O(E)$ are those fulfilling the condition ${}^t SAS = A$. In particular one always has $\det f = \pm 1$ and as usual we put

$$O^\pm(E) = \{ f \in O(E) : \det f = \pm 1 \}.$$

The *involutions* of $O(E)$ are exactly the maps of the form $f = \mathrm{Id}_S - \mathrm{Id}_T$, where

$E = S \oplus T$ is an *orthogonal* direct sum; according to 13.D they correspond exactly to non-singular subspaces S. Involutions are also called *symmetries* or *reflections* (through S).

As in the Euclidean case, we can show that every element of $O(E)$ is the product of at most n reflections through hyperplanes, but the proof in this case is much more difficult (Cartan-Dieudonné); see problem 13.5. One essential result is Witt's theorem: let F, F' be two subspaces of E and f be a linear map from F into F' such that $q|_F = f^*(q|_{F'})$. Then f can be extended to an element of $O(E)$.

A useful tool for the results above is the following: let F be a subspace of E, and suppose its radical $\operatorname{rad}(F)$ has dimension s. Let G be such that $G + \operatorname{rad}(F) = F$, and let $\{x_i\}_{i=1,\dots,s}$ be a basis for $\operatorname{rad}(F)$. Then there are s planes P_i in E such that each P_i contains x_i and is an Artin space under $q|_{P_i}$ (see 13.B), the P_i and G are pairwise orthogonal, and the orthogonal direct sum $\bar{F} = G \oplus P_1 \oplus \cdots \oplus P_s$ is non-singular (we say that $\bar{F}$ is a *non-singular completion* of F). This lemma shows immediately that the null subspaces of E have dimension at most $n/2$; moreover, if E has a null subspace of dimension $s = n/2$, then E is necessarily an Artin space Art_{2s}. See also problem 13.1.

The Witt theorem shows in addition that the maximal null subspaces of a pair (E,q) are all conjugate under elements of $O(E)$, and in particular have the same dimension.

13.F The Two-dimensional Case ([B, 13.8])

In dimension 2, the subgroup $O^+(E)$ is always commutative, and $O^-(E)$ is formed by reflections through lines; see problem 13.4.

In E there are either two distinct isotropic lines or no isotropic lines; in the first case we necessarily have $E = \mathrm{Art}_2$, and we can study $O(E)$ in the same way we did for the Euclidean case (cf. [B, 13.8]).

Problems

13.1 ANISOTROPIC FORMS ([B, 13.9.4]). Reduce the classification of quadratic forms to that of anisotropic forms.

13.2 FORMS IN DIMENSION 1 OVER A FINITE FIELD ([B, 13.9.10]). Show that if $n=1$ and K is finite, there are exactly three classes of quadratic forms.

13.3 FORMS IN DIMENSION 1 OVER THE RATIONALS ([B, 13.9.9]). Show that if $K = \mathbf{Q}$ and $n = \dim E = 1$ there are an infinite number of non-isometric forms (E,q).

13.4 AN EXCEPTIONAL PLANE ([B, 13.9.16]). Show that $O(E)$ is never commutative unless $E = \mathrm{Art}_2$ over the field with three elements.

13.5 EXCEPTIONAL ISOMORPHISMS ([B, 13.9.15]). Show there are vector spaces E possessing quadratic forms q such that (E, q) admits isomorphisms f (i.e. $f \in O(q)$) satisfying the following condition: $f(x) - x$ is non-zero and isotropic for any non-isotropic vector x.

Chapter 14

Projective Quadrics

In this chapter E denotes a vector space of dimension $n+1$ over a commutative field K of characteristic different from 2; the associated projective space is *denoted* by $P(E)$, and $p: E\setminus 0 \to P(E)$ is the canonical projection. We *denote* by $Q(E)$ the vector space of quadratic forms over E, and by $\mathrm{PQ}(E)$ the associated projective space $P(Q(E))$; the polar form of $q \in Q(E)$ is *denoted* by P. We will always have $n \geq 1$.

14.A Definitions ([B, 14.1])

A (*projective*) *quadric* in $P(E)$ is a non-zero element α of $\mathrm{PQ}(E)$, i.e. a quadratic form q, over E, considered up to a non-zero scalar. Such a form q representing the class α is called an *equation* of α.

The *image* of α, *denoted* by $\mathrm{im}(\alpha)$, is defined as $p(q^{-1}(0)\setminus 0)$; it is the image in $P(E)$ of a cone of E, and it may be empty. The *rank* of α is the rank of one of its equations; the quadric is called *degenerate* if q is degenerate, and *proper* otherwise. When $n = 2$ we use the term *conic* instead of quadric.

14.B Notation, Examples ([B, 14.1])

If α is a quadric in $P(E)$ and $S = P(F)$ is a (projective) subspace of $P(E)$, we define the *intersection* of α and S as the quadric having $q|_F$ as an equation (where q is an equation of α); *denoting* this intersection by $\alpha \cap S$ we have,

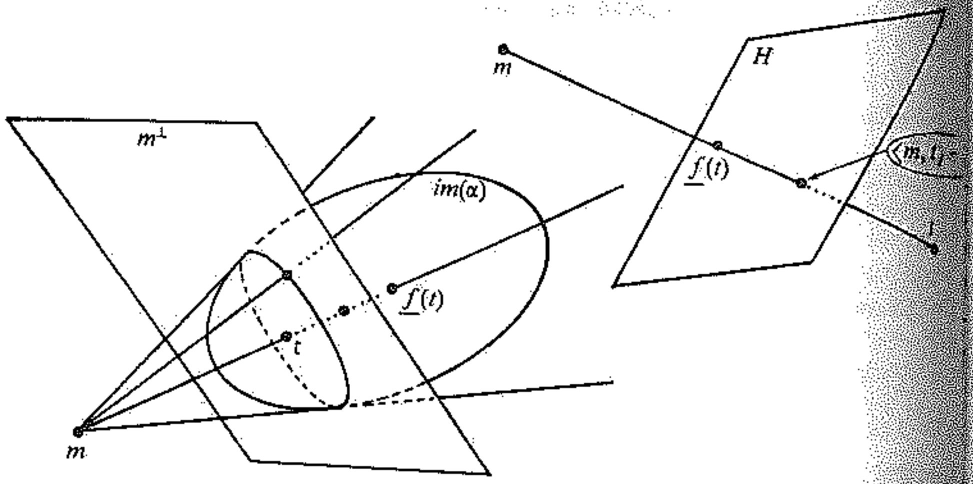


Figure 14.G.

Problems

14.1 THE COMPLEX QUADRIC AND THE GRASSMANN MANIFOLD ([B, 14.8.4]. Let $C(n)$ be the complex quadric in dimension n (cf. 14.C), i.e. the unique non-degenerate quadric of the complex projective space $P^n(\mathbf{C})$ given in homogeneous coordinates by the equation $\Sigma_{k=0}^{n} z_k^2 = 0$.

Show that $C(n)$ is homeomorphic to the Grassmann manifold of oriented lines of $P^n(\mathbf{R})$, which is the same as the set of oriented two-dimensional vector subspaces of $\mathbf{R}^{n+1}$.

14.2 SIX POINTS ON THE SAME CONIC ([B, 14.8.11]). Let α be a proper conic, $\{a, b, c\}$ and $\{a', b', c'\}$ two self-polar triangles with respect to α; show that the six points a, b, c, a', b', c' belong to one single conic.

14.3 HARMONICALLY INSCRIBED QUADRIC ([B, 14.8.10]). We say that the proper quadric α' is *harmonically inscribed* in α if trace $(\varphi'^{-1}\varphi) = 0$. Interpret this condition geometrically.

Deduce that two triangles which are self-polar with respect to the same conic are circumscribed around one single conic. Prove also that if there is a triangle inscribed in a conic C and circumscribed around a conic γ, every point of C from which it is possible to take tangents to Γ is the vertex of a triangle inscribed in C and circumscribed around Γ.

Show finally that the circle circumscribed around a triangle circumscribed around a parabola passes through its focus.

Chapter 15

Affine Quadrics

In all of this chapter X will be an affine space of finite dimension $n \geq 1$ over a commutative field of characteristic $\neq 2$. We shall use (cf. chapter 5) the projective completion $\tilde{X} = X \cup \infty_X$ of X, where the hyperplane at infinity is $\infty_X = P(\vec{X})$.

15.A Definitions ([B, 15.1])

An *affine quadratic form* over X is a polynomial over X whose degree is less than or equal to 2 (cf. 3.E); we *denote* by $Q(X)$ the vector space of such polynomials. The *symbol* $\vec{q}$ of $q \in Q(X)$ is a polynomial of degree 2 over $\vec{X}$. In every vectorialization of X, we can write $q = q_2 + q_1 + q_0$, where $q_0 \in K$, q_1 is a linear form, and $q_2 = \vec{q}$.

An (affine) *quadric* in X is an element α of the projective space $\mathrm{QA}(X) = P(Q(X))$ such that, for $\alpha = p(q)$, we have $\vec{q} \neq 0$. If $\alpha = p(q)$, we say that q is an *equation* of α; if $n = 2$ we use the term *conic* instead of *quadric*. The *image* of α is $\mathrm{im}(\alpha) = q^{-1}(0)$.

By passing from X to $\tilde{X}$, we see that there is a bijection $\tilde{}$ between the affine quadrics of X and the projective quadrics $\beta = \tilde{\alpha}$ of $\tilde{X}$ such that $\mathrm{im}(\beta)$ does not contain ∞_X. Under this correspondence it is true that $\mathrm{im}(\alpha) = \mathrm{im}(\tilde{\alpha}) \cap X$ and $\tilde{\alpha} \cap \infty_X = \vec{\alpha}$ if $\vec{\alpha} = p(\vec{q})$ for an equation q of α.

We say that α is *proper* if $\tilde{\alpha}$ is; the *rank* of α is that of $\tilde{\alpha}$ and the *index* of $\vec{\alpha}$ is the rank of α.

Expressed in an affine frame, an equation q of α will be:

$$\sum a_{ij} x_i x_j + 2 \sum b_i x_i + c, \quad \text{where} \quad \vec{q} = \sum a_{ij} x_i x_j. \tag{1}$$

15.C Polarity ([B, 15.5])

Polarity relative to a *proper* affine quadric α is by definition the same as polarity relative to $\tilde{\alpha}$; it is a relation in X, or if necessary in $X \cup \infty_X$ (cf. 14.E and chapter 5).

It is easy to see that the following three conditions are equivalent: the quadric α is of type II, the hyperplane ∞_X is not tangent to X, and the pole $C = \infty_X^{\perp}$ of the hyperplane at infinity is not at infinity. In this situation we say that α is a *central quadric*, for in effect $c = \infty_X^{\perp}$ is a center of symmetry for $\operatorname{im}(\alpha)$.

If $(\xi_1, \ldots, \xi_n)$ is a point in X, its polar hyperplane relative to the quadric of equation (1) (15.A) has the following equation (cf. 14.E):

$$\sum_{i,j} a_{ij}\xi_i x_j + \sum_i b_i(\xi_i + x_i) + c = 0.$$

The equations of the center $(x_1, \ldots, x_n)$ are then

$$\sum_j a_{ij}x_j + b_i = 0 \quad (i = 1, \ldots, n).$$

An interesting case of polarity is when we take a point $a \in \infty_X$; if α is a central quadric, the polar hyperplane $a^{\perp}$ of a is an affine hyperplane passing through the center c, and the affine reflection through the hyperplane $a^{\perp}$ and parallel to the direction $\dot{a}$ (cf. 2.D) leaves invariant the image of α. We say that $a^{\perp}$ is a *diametral* hyperplane for α; for $n = 2$ we call it a *diameter*. If we take n points $a_1, \ldots, a_n$ in ∞_X such that the simplex $\{c, a_1, \ldots, a_n\}$ is self-polar relative to α (cf. 14.E) (which here means that the points a_i are pairwise conjugate with respect to α), then the lines going through c and whose directions are the a_i are said to form a set of *conjugate diameters* of α.

We extend the notion of diameter to quadrics of type III (typified by the parabola) in the following way: instead of passing through the center, the diameters are all parallel and their direction is that of the point at infinity of α.

15.D Euclidean Affine Quadrics ([B, 15.6])

Because of 13.B, every proper affine quadric in a Euclidean affine space will have, in some appropriate orthonormal frame, one of the following equations:

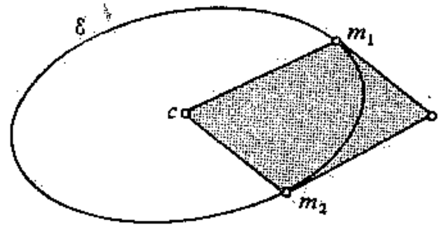
$$\sum_{i=1}^{r} a_i x_i^2 - \sum_{i=r+1}^{n} a_i x_i^2 + 1 \quad \text{or} \quad \sum_{i=1}^{r} a_i x_i^2 - \sum_{i=r+1}^{n-1} a_i x_i^2 + 2x_n,$$

where all the a_i are strictly positive. The quadric is called an *ellipsoid* if $\operatorname{im}(\alpha) = \mathscr{E}$ can be written as

$$\mathscr{E} = \mathscr{E}(q) = \{x \in X : q(x) = 1\}$$

where q is a positive definite quadratic form over the vectorialization X_a, and a is the center of $\mathscr{E}$.

The *theorem of Appolonius* says the following: let $\mathscr{E}$ be an ellipsoid with center a, and let $\{m_i\}_{i=1,\ldots,n}$ be a set of points of $\mathscr{E}$ such that the directions $\overrightarrow{am_i}$ $(i = 1, \ldots, n)$ are conjugate (which means that the lines $\langle a, m_i \rangle$ form a set of conjugate diameters of $\mathscr{E}$, in the sense of 15.C). Then, for each k, the sum $\sum_{i_1 < \cdots < i_k} \operatorname{Gram}(m_{i_1}, \ldots, m_{i_k})$ of the Gram determinants of all the k-element subsets of the m_i is a constant, depending only on q and not on the choice of the m_i. The cases $k = 1$ and $k = n$ are particularly interesting; see problems 15.3 and 15.4, and 17.D.2.



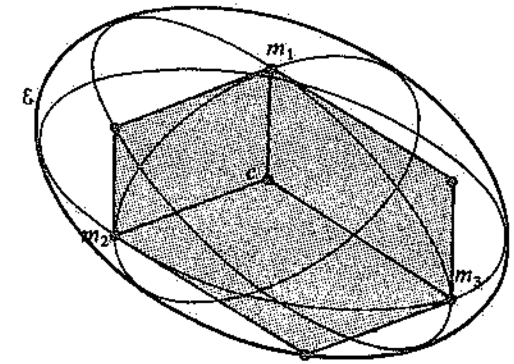


Figure 15.D.

Problems

15.1 ARCHIMEDES' METHOD FOR THE AREA OF THE PARABOLA ([B, 15.7.6]). Let $C = \operatorname{im}(\alpha)$ be the non-empty image of a proper plane conic, let m be a point on the plane, and $\langle m, a \rangle$ and $\langle m, b \rangle$ two distinct tangents to diameter of α, and that the tangents to α at the points of $C \cap D$ are parallel to $\langle a, b\rangle$. When α is a parabola, show that D always intersects C, and the intersection is the midpoint of m and $(a+b)/2$. Deduce a geometric construction for a sequence of points on an arc of parabola, given two points and the tangents at these points.

Observing that the area of the triangle $\{m, a', b'\}$ in figure 15.1 is 1/4 of the area of $\{m, a, b\}$, deduce that the shaded area is 2/3 of the area of $\{m, a, b\}$ (this limiting process is due to Archimedes).

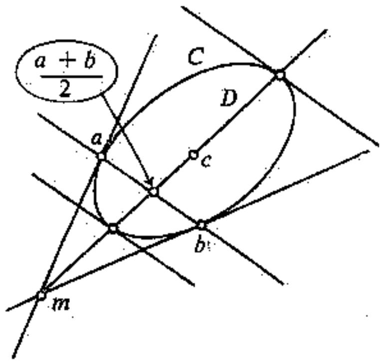


Figure 15.1.1.

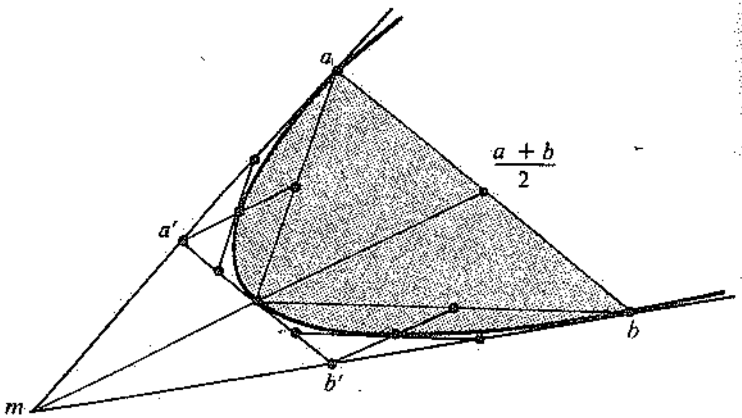


Figure 15.1.2.

15.2 ELLIPSES AND PARALLELOGRAMS Given a parallelogram, show that there are ellipses inscribed in them so that the tangency points are at the middle of each side. With the notation of figure 15.2.1, show that such an ellipse always satisfies $\overrightarrow{c\alpha} = \sqrt{2}\,\overrightarrow{c\beta}$.

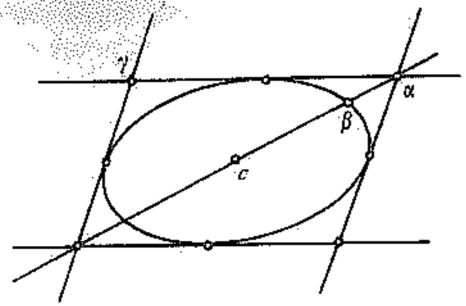


Figure 15.2.1.

15.3 METRIC RELATIONS IN ELLIPSOIDS: I ([B, 15.7.9]). Let Q be the image of an ellipsoid in a three-dimensional Euclidean affine space, and let x be a fixed point in the interior of Q (cf. 15.D). Take three orthogonal lines D, E, F through x, and let the intersection points of Q with D be a, b, with E be c, d and with F be e, f. Show that the sum

$$\frac{1}{\overline{xa}\cdot\overline{xb}} + \frac{1}{\overline{xc}\cdot\overline{xd}} + \frac{1}{\overline{xe}\cdot\overline{xf}}$$

is constant. Give examples and generalize.

Now consider three lines D, E, F through x whose directions are pairwise conjugate relative to Q, and let the intersection points of Q with D be a, b, with E be c, d and with F be e, f. Show that the sum

$$\overline{xa}\cdot\overline{xb} + \overline{xc}\cdot\overline{xd} + \overline{xe}\cdot\overline{xf}$$

is constant.

15.4 METRIC RELATIONS IN ELLIPSOIDS: II ([B, 15.7.20]). Let $\mathscr{E}$ be an ellipsoid with center O in an n-dimensional Euclidean affine space (cf. 15.D). We consider the sets $\{a_i\}_{i=1,\ldots,n}$ of points of $\mathscr{E}$ such that the vectors $\overrightarrow{Oa_i}$ are orthogonal. Show that

$$\sum_{i=1}^{n} \frac{1}{(Oa_i)^2}$$

is a constant.

Use this fact to find the envelope of the hyperplanes containing the a_i.

Using polarity with respect to a sphere centered at O (cf. 10.B), show that the preceding result implies that the locus of the points which are the intersection of n orthogonal hyperplanes tangent to an ellipsoid is a sphere (called the *orthoptic sphere* of that ellipsoid).

15.5 NORMALS TO A QUADRIC FROM A GIVEN POINT ([B, 15.7.15]). Let Q be a quadric in a 3-dimensional Euclidean affine space, and let m be a point. Show that the number of *normals* to Q that pass through m is "in general" equal to six. Show that the feet of all normals to Q from m are contained in a second-degree cone with vertex m and containing the center of Q and the parallels to the axes of Q which go through m.

15.6 HOMOFOCAL QUADRICS ([B, 15.7.17]). We consider in $\mathbf{R}^3$ the family of quadrics $Q(\lambda)$ whose equations are

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-1=0,$$

with $a>b>c$. Find how many quadrics $Q(\lambda)$ pass through a fixed point (x_0, y_0, z_0); show that if three quadrics $Q(\lambda)$ pass through a point, their tangent planes at that point are orthogonal (cf. 17.B, 17.D).

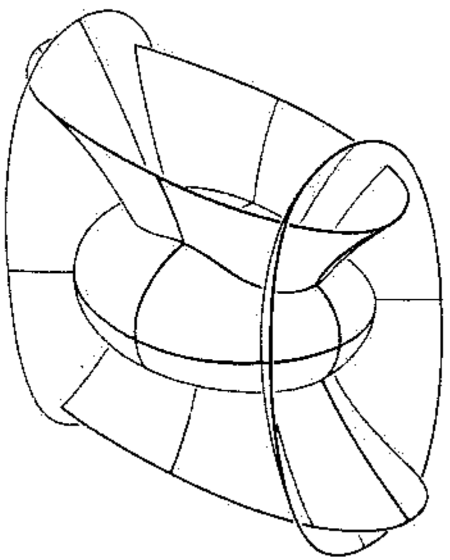


Figure 15.6.

Chapter 16

Projective Conics

In all of this chapter, $P=P(E)$ is a projective plane over a commutative field K of characteristic $\neq 2$; we *put* $P^*=P(E^*)$. We will often identify a point $m\in P$ with its homogeneous coordinates (x,y,z).

We will generally fix a conic $\alpha\in \mathrm{PQ}(E)$ and its image $C=\mathrm{im}(\alpha)$ (in most cases, α will be proper and have non-empty image), as well as one equation q for α. For a a point of C, the tangent to C at a will sometimes be *denoted* by $\langle a,a\rangle$.

16.A Notation ([B, 16.1])

The general equation of a conic will be written

$$q=ax^2+a'y^2+a''z^2+2byz+2b'zx+2b''xy.$$

Depending on whether the triangle $p(1,0,0)$, $q(0,1,0)$, $r(0,0,1)$ is inscribed in C, self-polar relative to C (cf. 14.E) or "bitangent to C", we have three simplified equations for the conic shown in the figure below:

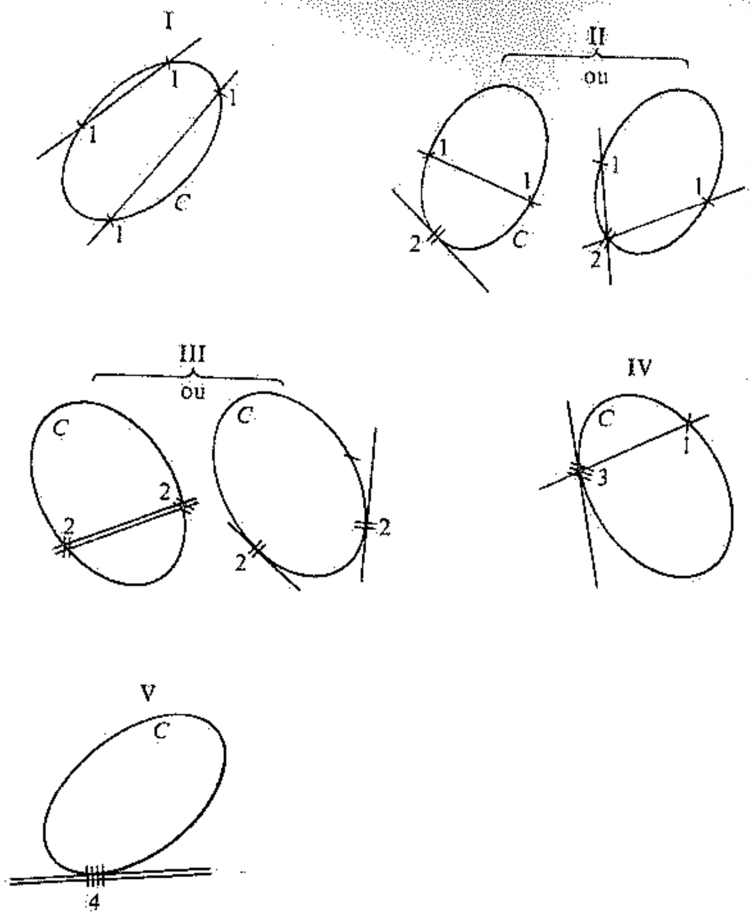


Figure 16.E.

16.F Pencils of Conics ([B, 16.5])

Recall (cf. 14.D) that a pencil of conics is a line in $PQ(E)$, or, analytically, the set of conics with equations $\lambda q + \lambda' q'$, where q, q' are two equations of conics (and we assume that at least one of them is proper), and (λ, λ') runs through $\tilde{K}$. In the algebraically closed case, a pencil of conics is the set of conics that intersect a given proper conic C in a fixed set $\{(m_i, \omega_i)\}$ of points of C satisfying $\Sigma_i \omega_i = 4$. So there are five types, I, II, III, IV and V, of pencils (see drawings in [B, 16.5]).

The degenerate conics in these pencils are: three pairs of lines in case I, two pairs of lines in case II, one pair of lines and a double line in case III, one pair of lines in case IV, and a double line in case V; see figure 16.E.

16.G Tangential Conics

Recall 14.F. We will have here five types I*, II*, III*, IV* and V* of tangential pencils of conics. Geometrically, III = III* and V = V* (bitangent and superosculating conics. For case I*, see problem 16.6.) Recall that the matrix that gives the envelope equation is the inverse of that for the punctual equation.

16.H The Great Poncelet Theorem ([B, 16.6])

This is a delicate theorem, and its proof is involved. It applies in the algebraically closed case, and it says that if C and Γ are two conics such that there is an n-sided polygon, all of whose vertices are on C and all of whose edges are tangent to Γ, then there are infinitely many such polygons, and one of the vertices can be arbitrarily chosen on C. See simple particular cases in problems 10.2, 14.3 and 16.5.

16.I Affine Conics ([B, 16.7])

The equation of an affine conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

The conic is proper if

$$\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix} \neq 0.$$

Its points at infinity are the lines of slope ϑ, where ϑ satisfies the equation

$$a + 2b\vartheta + c\vartheta^2 = 0$$

Problems

16.1 TRIANGLE CIRCUMSCRIBED AROUND A CONIC ([B, 16.8.2]). Show that if a, b, c is a triangle circumscribed around C, and α, β, γ are the tangency points, then the segments $a\alpha$, $b\beta$, $c\gamma$ are concurrent. Use only analytic geometry in your proof.

16.2 THEOREM OF PASCAL ([B, 16.8.5]). Let C be any conic and a, b, c, d, e, f any six points on C. Then the points $\langle a, b\rangle \cap \langle d, e\rangle$, $\langle b, c\rangle \cap \langle e, f\rangle$, $\langle c, d\rangle \cap \langle f, a\rangle$ are collinear. Prove this result using calculus, by taking a projective base formed by four of the six points considered.

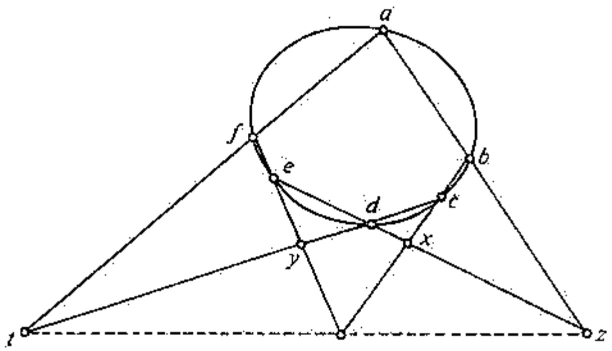


Figure 16.2.1.

16.3 CROSS-RATIOS FOR A CONIC ([B, 16.8.6]). Let C be the non-empty image of a proper conic, and let p, q, r be such that C is tangent to pq at q and to pr at r. Show that, for any $m, n \in C$, the following holds (cf. 16.C):

$$[q, r, m, n]_C^2 = [pq, pr, pm, pn].$$

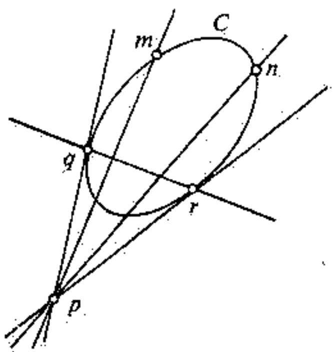


Figure 16.3.1.

16.4 COMMUTING INVOLUTIONS ([B, 16.8.7]). Show that two involutions of a proper conic whose image is non-empty commute if and only if their Frégier points are conjugate (cf. 16.D).

16.5 THE GREAT PONCELET THEOREM FOR BITANGENT CONICS

tangent to C, and a scalar k such that $q' = q + kd^2$, where q, q', d are equations of C, C', D, respectively (cf. 16.E). Notice that if the field K is closed, C and C' are indeed tangent at two distinct points.

Suppose the field K is **R** or **C** (if not, we must use an algebraic closure of K). Let f be a non-involutive homography (different from the identity) of a proper conic C. Show that the set of lines $\langle m, f(m)\rangle$, for m ranging through C, is the set of tangents to a proper conic which is bitangent to C in the sense above. Prove a converse statement. For two such bitangent conics, prove the great Poncelet theorem (16.H).

16.6 TANGENTIAL PENCILS OF CONICS ([B, 16.8.10]). The figure below shows several conics belonging to the same pencil. Prove rigorously that there are regions of the plane which do not intersect any of the conics of the pencil.

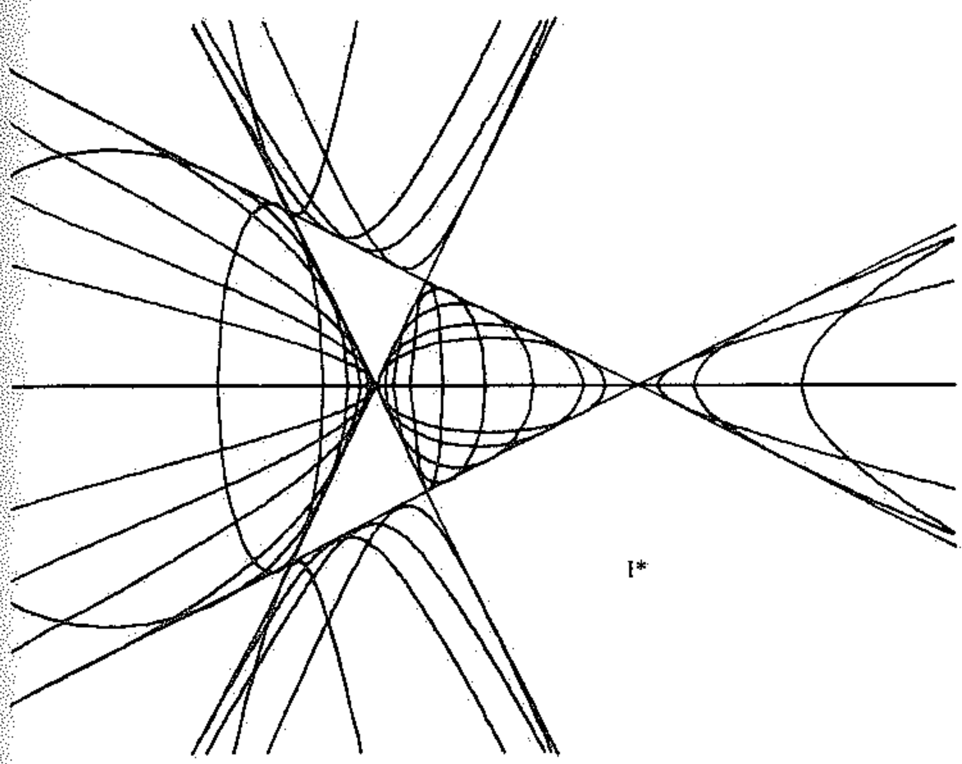


Figure 16.6.1.

16.7 INTERSECTION OF TWO CONICS OVER A FINITE FIELD ([B, 16.8.16]). Study the intersection of $xz - y^2 = 0$ and $xy - z^2 = 0$ over the field K with three elements hyperbola C, the two tangents and the lines $\langle m, f\rangle, \langle m, f'\rangle$ have the same bisectors.

17.C Using the Cyclical Points ([B, 17.4, 17.5])

We *denote* by $\bar{\alpha}$ the projective conic in $\tilde{X}^C$ obtained by complexifying the projective completion $\tilde{\alpha}$ of the conic α (cf. 15.A). On the line at infinity ∞_X^C, the position of the points at infinity of $\bar{\alpha}$ can take interesting special values relative to the cyclical points $\{I, J\}$ (cf. 9.D):

— α is an equilateral hyperbola if and only if its points at infinity are conjugate relative to $\{I, J\}$;

— α is a circle if and only if its points at infinity are $\{I, J\}$ themselves.

We can see immediately that the Laguerre formula (8.H) and the fact that the cross-ratio in 16.C is constant imply the condition we found in 10.D for four points to be on the same circle. See also problem 16.3.

Combining the above with the theorem of Desargues for pencils of conics (cf. 14.D), taking D to be the line at infinity of X, one obtains a number of results concerning pencils of Euclidean conics. For instance:

— the directions of the (symmetry) axes of a conic C are the points ∞_X^C which are harmonic conjugates relative to $\{I, J\}$ and also relative to the points at infinity of C;

— a pencil of conics contains a single equilateral hyperbola, unless it contains only such curves;

— a pencil of conics contains a circle if and only if the directions of the axes are fixed; in particular, the common chords have the same inclination relative to the axes, and the axes of the two parabolas in the pencil are orthogonal;

— the centroid of four cocyclical points on a parabola is located on its axis.

The foci of C are the points through which the tangents to C contain the cyclical points: cf. [B, 17.4.3].

17.D Notes

1. The set of plane conics (ellipses and hyperbolas) which share two common foci f, f' satisfies a number of properties. It forms a tangential pencil (cf. 14.F). For details, see [B, 17.6.3], and also problem 15.6.

2. The theorem of Appolonius (cf. 15.D), applied to the special case of conics, says that the area and the sum of the squares of the sides of a parallelogram

Problems

17.1 CHORDS OF A CONIC ([B, 17.9.20]). Given a fixed point on a conic, consider all the chords whose angle, seen from that point, is a constant. Find the envelope of these chords. Analyze the special case of the right angle.

17.2 COCYCLIC POINTS AND NORMALS TO AN ELLIPSE ([B, 17.7.3, 17.9.15, 17.9.10]). Consider the ellipse parametrized by $(a\cos t, b\sin t)$, where the parameter t is defined modulo 2π, and put $\vartheta = \tan t/2$.

(i) Show that the four points corresponding to values of the parameter $(t_i)_{i=1,2,3,4}$ are on the same circle if and only if

$$t_1 + t_2 + t_3 + t_4 \equiv 0 \pmod{2\pi}.$$

(ii) Consider four points $(m_i)_{i=1,2,3,4}$ on an ellipse, and take the points n_i where the osculating circles at m_i intersect the ellipse (we choose $n_i \neq m_i$ except in the superosculating case). Show that if the m_i are cocyclic, then so are the n_i.

(iii) Show that the normals to the ellipse through the points parametrized by $(t_i)_{i=1,2,3,4}$ are concurrent if and only if the corresponding ϑ_i satisfy the following two conditions:

$$\vartheta_1\vartheta_2 + \vartheta_1\vartheta_3 + \vartheta_1\vartheta_4 + \vartheta_2\vartheta_3 + \vartheta_2\vartheta_4 + \vartheta_3\vartheta_4 = 0,$$
$$\vartheta_1\vartheta_2\vartheta_3\vartheta_4 = -1.$$

In this case we have $t_1 + t_2 + t_3 + t_4 \equiv \pi \pmod{2\pi}$.

(iv) If four points in an ellipse have concurrent normals, then the circle passing through three of them also passes through the point diametrically opposite to the fourth (theorem of Joachimstal).

17.3 TANGENT CIRCLES TO TWO CONJUGATE DIAMETERS ([B, 17.9.21]). Show that the tangent circles to two variable conjugate diameters of an ellipse, and whose center is on the ellipse, have constant radius.

17.4 TANGENT ELLIPSES TO A CIRCLE ([B, 17.9.23]). Given a circle C in X and two points a, b on C, consider the ellipses E which are tangent to C, pass through a and b, and whose center is the midpoint of ab. Show that all such ellipses have the same excentricity.

17.5 NORMALS FROM A POINT TO A PARABOLA ([B, 17.9.18.2]). Show that the normals at three points m, m', m'' of a parabola P are concurrent if and only if the barycenter $(m + m' + m'')/3$ belongs to the axis of P. Also if and only if the circle passing through m, m', m'' contains the vertex of P.

We can obtain the circles of Villarceau in an elementary fashion by cutting the torus by oblique bitangent planes. We can also utilize the theory of cyclids: see problems 18.7 and 20.C, as well as [B, 20.7.2].

18.E The Möbius Group ([B, 18.10])

The spheres S^d is a homogeneous space under a group (cf. 1.B) strictly bigger than $O(d+1)$ (cf. 18.B), which is a Lie group of dimension $[d(d+1)]/2$. This larger group, called *conformal group* or *Möbius group* of S^d and *denoted* by $Möb(d)$, has dimension $[(d+1)(d+2)]/2$. It can be obtained in several equivalent ways.

First, it is the group of conformal transformations of S^d. It is also the group of transformations of S^d generated by $O(d+1)$ and the pull-backs under stereographic projection of the vector homotheties of $\mathbf{R}^d$. Also the group of transformations of S^d which transform every subsphere into a subsphere. Also the group formed by the restrictions to S^d of all inversions (cf. 10.C) and reflections through hyperplanes in $\mathbf{R}^{d+1}$ which leave S^d invariant. Finally, it can be identified with the projective group of the quadric α with equation $-\sum_{i=1}^{d+1} x_i^2 + x_{d+2}^2$ (cf. 14.G) in $P(\mathbf{R}^{d+2})$; the idea is to homogenize the affine equation $\sum_{i=1}^{d+1} x_i^2 = 1$ of S^d (see also chapter 15).

Problems

18.1 THE SPHEROMETER ([B, 18.11.1]). Let A denote the three vertices of an equilateral triangle of side a, B a point of the perpendicular to the triangle passing through its center, and e the distance from B to the plane defined by the triangle. Show that the radius of the sphere passing through B and the three points A has the value $R = (a^2 + 3e^2)/6e$.

One can build a device to measure the radius of a spherical surface by using the formula above. In the case of the device shown in figure 18.1.2, explain the function of the lever system at the top of the spherometer.

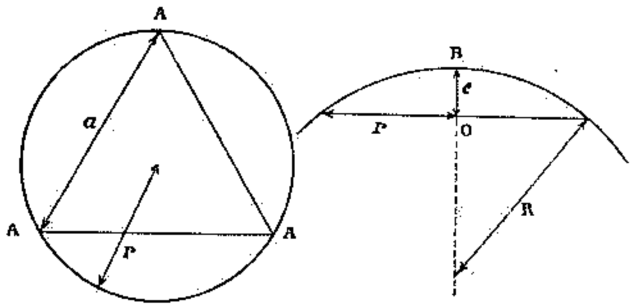


Figure 18.1.1.

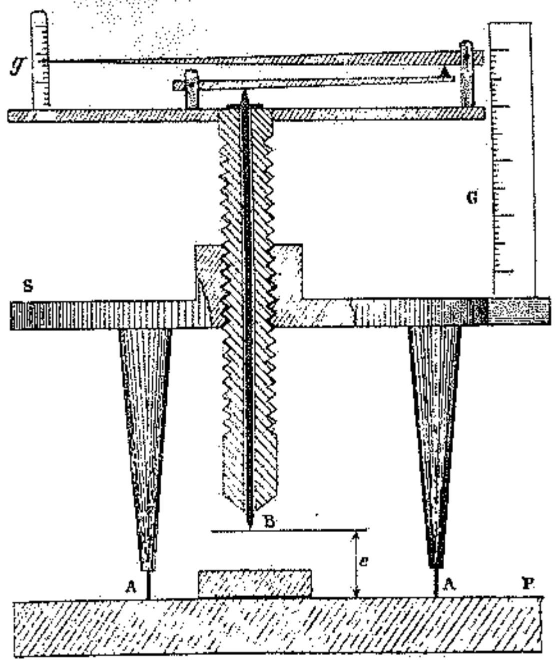


Figure 18.1.2.

H. Bouasse, Appareils de mesure, Delagrave, 1917.

18.2 LOXODROMES ([B, 18.11.3]). Recall that a loxodrome, or rhumb line, of the terrestrial sphere is a curve that makes a constant angle with the meridian at each point. (In the projection of Mercator, loxodromes become straight lines; they represent the trajectories of a ship whose helm is kept fixed. See [B, 18.1.8.2].) Show that, using stereographic projection centered at the north pole (18.A), loxodromes become logarithmic spirals (cf. 9.E). See also problem 6.6.

18.3 THE STRICT TRIANGLE INEQUALITY HOLDS FOR THE SPHERE ([B, 18.11.13]). Prove, using Gram determinants (cf. 8.J), that the distances a, b, c between any three points x, y, z in the sphere verify the inequalities

$$|b-c| \leq a \leq b+c,$$

and equality can only take place if the three points are in the same plane (i.e. they are aligned on a great circle, cf. 18.A).

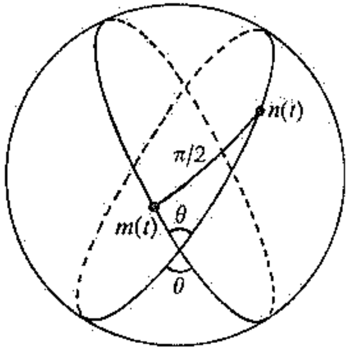
18.4 UNIVERSAL RELATION BETWEEN DISTANCES OF POINTS IN S^d ([B, 18.11.4]). Show that if $(x_i)_{i=1,\dots,d+2}$ are $d+2$ points in S^d, their distances $\overline{x_i x_j}$ always satisfy the relation

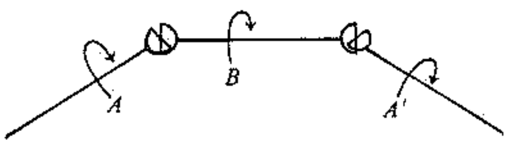
$$\det\left(\cos\left(\overline{x_i x_j}\right)\right) = 0.$$

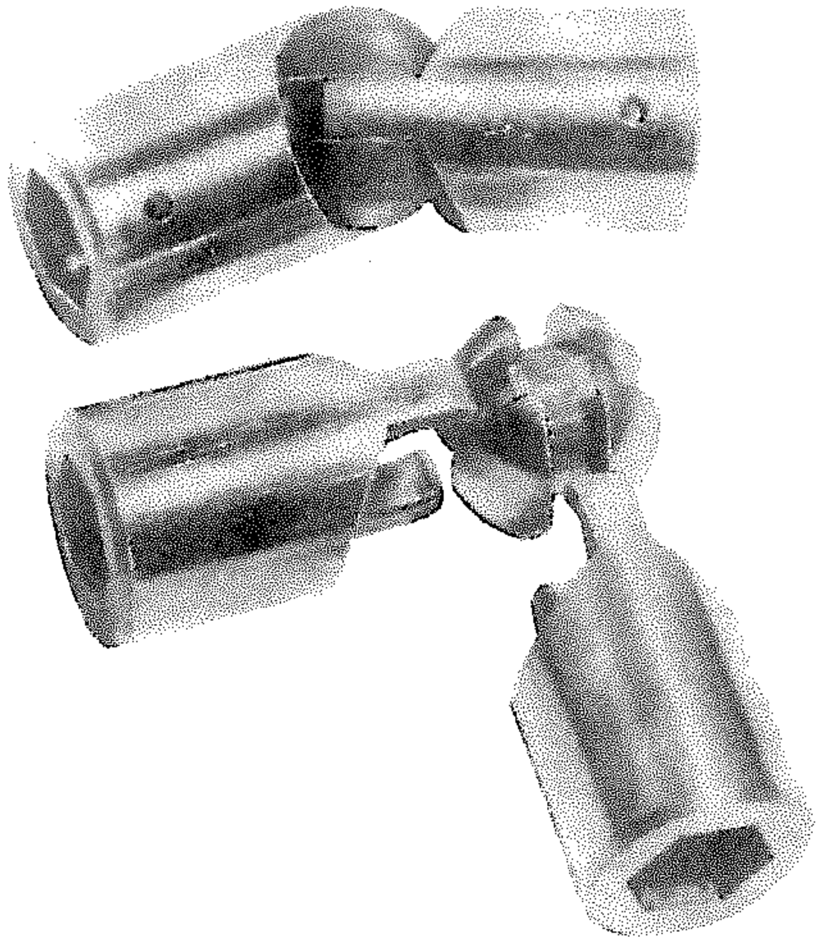
18.5 PLANE TRIGONOMETRY AS THE LIMIT OF SPHERICAL TRIGONOMETRY ([B, 18.11.9]). Generalize formulas (1), (2) and (3) of 18.C for the intrinsic metric of a sphere of radius R. Then find out what the formulas become when R approaches infinity.

18.6 HOOKE JOINTS, HOMOKINETIC JOINTS ([B, 18.11.16]). Consider a Hooke joint (figure 18.6.2) whose axes make an angle ϑ. The ratio between the instant angular velocities of the two shafts is a function of the angle between the plane of either fork and the plane containing the axes of the shaft; find the worst possible value for this ratio. To do this, take two great circles C, D in S^2 making an angle ϑ, and two moving points $m(t), n(t)$ on the circles, so that $\overline{m(t)\ n(t)} = \pi/2$. Find the value of the worst possible ratio when $\vartheta = \pi/3$, $\pi/4$, $\pi/6$.

Show that if two shafts A, A' are joined by Hooke joints to a third shaft B whose forks are in the same plane, in such a way that A, B, A' are in the same plane and the angles of B with A and A' are the same (*homokinetic joint*), then A and A' always have the same angular velocity.







18.7 DUPIN CYCLIDS ([B, 18.11.19]). Let $\Sigma, \Sigma', \Sigma''$ be three spheres in $\mathbf{R}^3$; show that, for certain configurations of these spheres, the set of spheres tangent to $\Sigma, \Sigma', \Sigma''$ has for envelope the surface obtained from a torus of revolution by inversion relative to an appropriate point. Deduce several properties of such surfaces, which we will encounter again in 20.C and which are called *Dupin cyclids*.

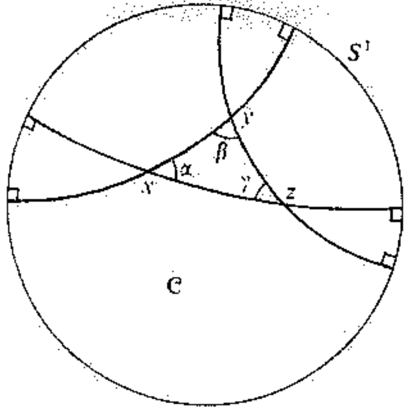


Figure 19.D.2.

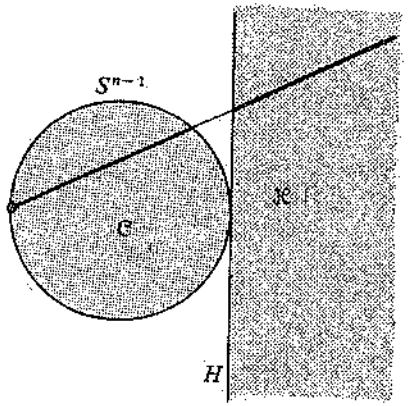


Figure 19.D.3.

Another model $\mathscr{H}$, called the *upper half-space model*, is obtained from $\mathscr{C}$ by applying inversion relative to a point on the boundary of $\mathscr{H}$; it is still conformal, and the segments are still circles.

Problems

19.1 ELLIPTICAL EQUILATERAL SETS ([B, 19.8.24]). An *equilateral set* of a metric space is any set $\{m_i\}_{i=1,\dots,n}$ such that all the distances $d(m_i, m_j)(i < j)$ are equal. Show that the elliptic plane P contains equilateral three-point sets with side lengths ranging from 0 to $\pi/2$; classify them under the action of $\mathrm{Is}(P)$. Show that P contains equilateral four-point sets with side length $\cos^{-1}(1/\sqrt{3})$ or $\cos^{-1}(1/\sqrt{5})$; study their behavior under the action of $\mathrm{Is}(P)$. Show that, up to isometries, P contains exactly one five-point and one six-point equilateral set, and their sides have length $\cos^{-1}(1/\sqrt{5})$.

19.2 HYPERBOLIC QUADRILATERALS WITH THREE RIGHT ANGLES ([B, 19.8.7]). Find the fourth angle of a quadrilateral in the hyperbolic plane such that three of its angles are right and the lengths of the sides which join two right angles are a and b.

19.3 UNIVERSAL DISTANCE RELATION IN HYPERBOLIC SPACES ([B, 19.8.16]). Show that in n-dimensional hyperbolic space every set of $n+2$ points $z_i (i = 1, \dots, n+2)$ obeys the relation

$$\det\left(\mathrm{ch}\left[d(z_i, z_j)\right]\right) = 0.$$

19.4 REGULAR HYPERBOLIC POLYGONS ([B, 19.8.20]). Here n is an integer ≥ 3. We want to study n-sided polygons in the hyperbolic plane, all of whose sides are equal and all of whose angles have the value $2\pi/n$. Are there such polygons for any n? Are they unique up to isometries?

Observe that true spheres are the points of $S(E)$ which lie "outside" $\hat{E}$. Those inside represent imaginary spheres (which, incidentally, are models for hyperbolic geometry; see 19.B).

Pencils of spheres are the (projective) lines of $S(E)$. They differ depending on whether the line does or does not intersect $\hat{E}$; see [B, 20.5.6].

The above construction allows us to extend inversion to the completion $\hat{E}$; the pole c is taken to ∞ and ∞ is taken to c. Inversions are in fact those elements of $PO(\rho)$ which are reflections through hyperplanes; this means they generate the conformal group $Möb(n)$ of S^n.

20.C Polyspheric Coordinates ([B, 20.7])

Since E is embedded in $S(E)$, we can represent the points of E using homogeneous coordinates in the projective space $S(E)$ (so there are $n+2$ coordinates for an n-dimensional space E); the chosen basis will preferably be one in which ρ is diagonal. Such coordinates are called *polyspheric*. *Cyclids* are the hypersurfaces of E which are quadrics when expressed in polyspheric coordinates; they are algebraic hypersurfaces of the fourth degree. The cyclids of Dupin are those which have two equal coefficients; the torus of revolution is a special case of a cyclid of Dupin. See problems 18.7 and 20.2. Certain cyclids contain six families of circles: [B, 20.8.7].

Problems

20.1 TANGENT HYPERPLANE TO $im(s)$ IN THE SPACE OF SPHERES ([B, 20.8.1]). Construct geometrically the tangent hyperplane to $im(s)$ in $S(E)$.

20.2 THE TORUS IS A CYCLID OF DUPIN ([B, 20.8.5]). Show that the torus is a cyclid of Dupin (cf. 20.C).

20.3 THE THEOREM OF DARBOUX ([B, 20.8.7]). If three points of a line describe three spheres whose centers are collinear, then every point of that line also describes such a sphere, or possibly a plane for one exceptional point. Find a relation between four points of the line and the centers of the four spheres they describe.

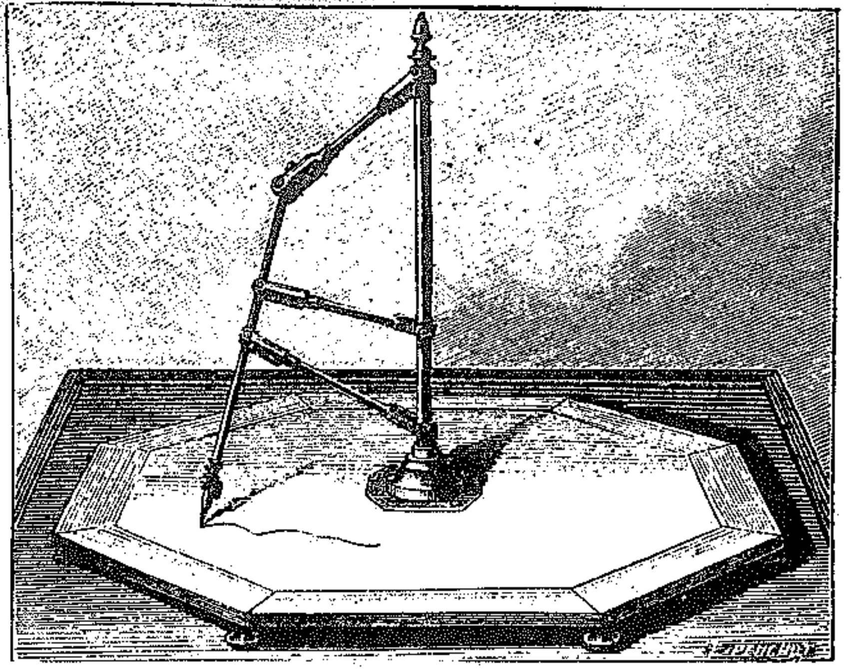


Figure 20.3.1.

Gabriel Koenigs, Leçons de cinématique, A. Hermann, 1897.