C^2-FINITE SEQUENCES

A Computational Approach

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ACA: D-finite Functions and Beyond

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C-finite sequences

Definition

A sequence $a(n) \in \mathbb{K}^{\mathbb{N}}$ is called C-finite if there are constants $\gamma_0, \dots, \gamma_r \in \mathbb{K}$, not all zero, such that

$$\gamma_0 a(n) + \cdots + \gamma_{r-1} a(n+r-1) + \gamma_r a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- Examples: Fibonacci numbers $f(n)$, Lucas numbers, Pell numbers, etc.
- The set of C-finite sequences forms a ring under termwise addition and multiplication.
- Every C-finite sequence can be described by finite amount of data.
- Every sequence $a(n)$ is bounded by α^n for some computable α, i.e., $|a(n)| \leq \alpha^n$ for all $n \geq 1$.

D-finite sequences

Definition

A sequence $a(n) \in \mathbb{K}^{\mathbb{N}}$ is called D-finite if there are polynomials $p_0(n), \ldots, p_r(n) \in \mathbb{K}[x]$, not all zero, such that

$$p_0(n)a(n) + \cdots + p_{r-1}(n)a(n+r-1) + p_r(n)a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- Also called: holonomic, P-recursive.
- Examples: C-finite sequences, factorial, Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.
- Set of D-finite sequences forms a ring.
- Every sequence is bounded by $(n!)^k$ for some computable $k \in \mathbb{N}$.

C^2-finite sequences

Definition

A sequence $a(n) \in \mathbb{K}^{\mathbb{N}}$ is called C^2-finite if there are C-finite sequences $c_0(n), \ldots, c_r(n) \in \mathbb{K}^{\mathbb{N}}$ with $c_r(n) \neq 0$ for all $n \in \mathbb{N}$ such that

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + c_r(n)a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- Contains C- and D-finite and q-holonomic sequences.
- Examples: $c(n^2)$ and $\prod_{k=0}^{n} c(k)$ for a C-finite sequence $c(n)$.
- Set of C^2-finite sequences forms a ring (Jiménez-Pastor, N., and Pillwein 2021).
- Every sequence is bounded by α^{n^2} for some α. It is not known, whether such an α can be computed.

Skolem-Problem

Skolem-Problem

Does a given C-finite sequence have a zero?

Not known whether decidable in general.

- Decidable for sequences of order ≤ 4 or if the sequence has a unique dominant root (Survey: Ouaknine and Worrell 2012).
- Sometimes the Gerhold-Kauers method using CAD can be applied (Gerhold and Kauers 2005).
- In practice, it is usually decidable (N. and Pillwein 2022a).

Simple C^2-finite sequences

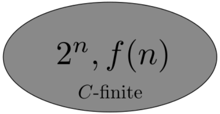
Definition

A sequence $a(n) \in \mathbb{K}^{\mathbb{N}}$ is called simple C^2-finite if there are C-finite sequences $c_0(n), \dots, c_{r-1}(n) \in \mathbb{K}^{\mathbb{N}}$ such that

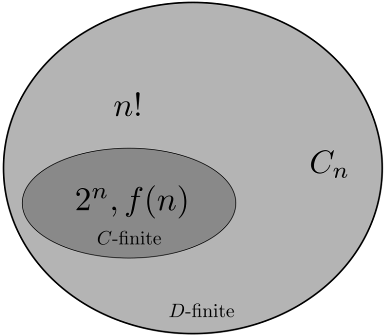
$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- Examples: $c(n^2)$ and $\prod_{k=0}^{n} c(k)$ for a C-finite sequence $c(n)$.
- Set of simple C^2-finite sequences forms a ring.
- The set of simple C^2-finite sequences over $\overline{\mathbb{Q}}$ is even a computable ring (N. and Pillwein 2022b).
- Every sequence is bounded by α^{n^2} for some computable α.

Overview: Rings of Sequences



Overview: Rings of Sequences



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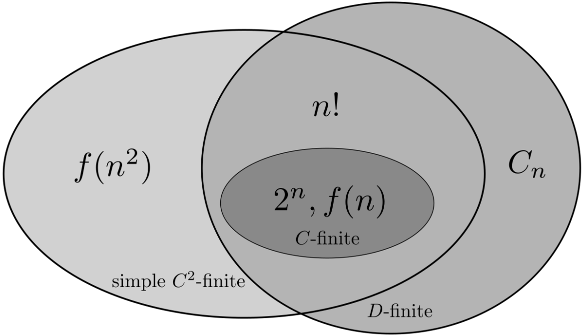
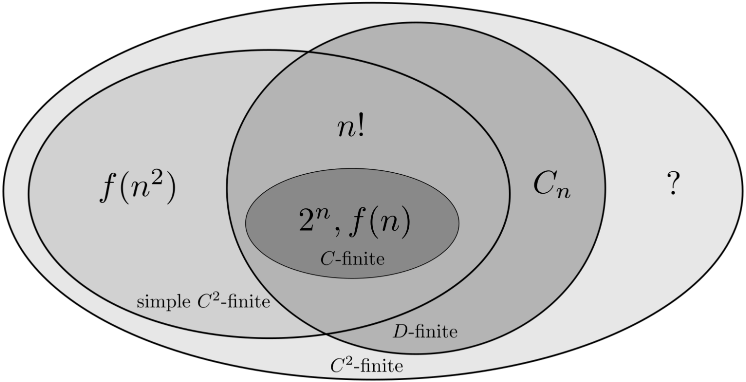
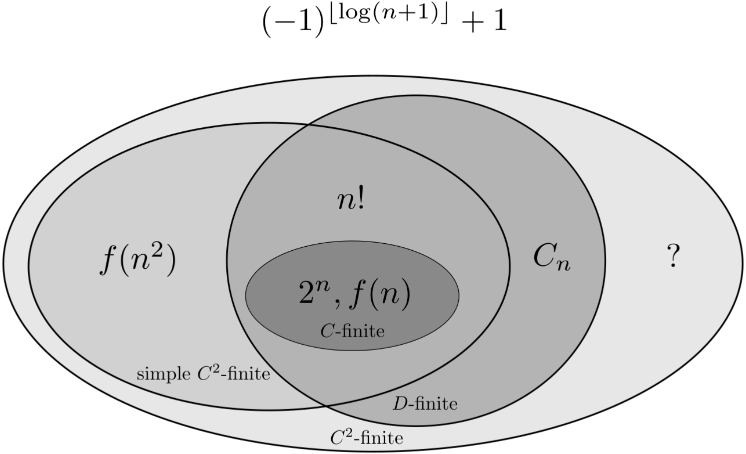


Overview: Rings of Sequences



Overview: Rings of Sequences



rec_sequences package

- rec_sequences is a SageMath package for linear recurrence sequences (mostly C-finite and C^2-finite sequences).
- Based on the ore_algebra package (Kauers, Jaroschek, and Johansson 2015).
- Can be obtained from GitHub: github.com/PhilippNuspl/rec_sequences

```
sage: from rec_sequences.CFiniteSequenceRing import *
sage: from rec_sequences.C2FiniteSequenceRing import *
```

A C-finite sequence can be defined by the recurrence and initial values or using guessing:

```
sage: C = CFiniteSequenceRing(QQ)
sage: fib = C([1, 1, -1], [1, 1])
sage: alt = C(10*[1 , -1])
```

Example: Sparse Fibonacci numbers

The sequence $f(n^2)$ is C^2-finite satisfying (Kotek and Makowsky 2014)

$$-f(2n+3)f(n^2) - f(4n+4)f((n+1)^2) + f(2n+1)f((n+2)^2) = 0.$$

```
sage: C2 = C2FiniteSequenceRing(QQ)
sage: sparse_fib = fib.sparse_subsequence(C2) # A054783
sage: var("n")
sage: coeffs = [-fib(2*n+3), -fib(4*n+4), fib(2*n+1)]
sage: sparse_fib.coefficients() == coeffs
True
sage: sparse_fib[:8]
[1, 1, 5, 55, 1597, 121393, 24157817, 12586269025]
```

Ring operations

Consider the fibonorial numbers $a(n) = \prod_{k=0}^{n} f(k)$ and the Pell-Lucas numbers $p(n)$ satisfying

$$f(n+1)a(n) - a(n+1) = 0, \qquad a(0) = 1,$$
$$p(n) + 2p(n+1) - p(n+2) = 0, \qquad p(0) = 2, p(1) = 2.$$

Then, the sequence $c = a + p$ is C^2-finite.

```
sage: fibonorial = C2([fib, -1], [1]) # A003266
sage: pell_lucas = C([1, 2, -1], [2, 2]) # A002203
sage: c = fibonorial + pell_lucas
sage: c.order()
3
sage: c[:100] == [fibonorial[n]+pell_lucas[n] for n in range(100)]
True
```

Ring operations

Consider the sequences

$$(-1)^n a(n) + a(n+1) = 0, \qquad a(0) = 1,$$
$$b(n) + b(n+1) = 0, \qquad p(0) = 1.$$

Then, the sequences ab and $a+b$ are C^2-finite.

```
sage: a = C2([C((-1)^n), 1], [1])
sage: b = C2([1, 1], [1])
sage: show(a*b)
```

$$(-(-1)^n) \cdot a(n) + (1) \cdot a(n+1) = 0 \quad a(0) = 1$$

```
sage: show(a+b)
```

$$\left(-\frac{1}{2}\,(-1)^n + \frac{1}{2}\right)\cdot a(n) + \left(\frac{1}{2}\,(-1)^n + \frac{1}{2}\right)\cdot a(n+2) + (1)\cdot a(n+3) = 0 \quad a(0) = 2, a(1) = -2, a(2) = 0$$

Note: The order bounds for C-finite sequences are not satisfied for $a+b$.

More closure properties

(Simple) C^2-finite sequences are also closed under

- partial sums,
- taking subsequences at arithmetic progressions,
- interlacing.

Example

The sequence $\sum_{k=0}^{\lfloor n/3 \rfloor} f((2k+1)^2)$ is C^2-finite.

```
sage: a = sparse_fib.subsequence(2, 1).sum().multiple(3)
sage: a.order()
9
```

Are (simple) C^2-finite sequences closed under the Cauchy product?
We do not know.

Further generalizations

- Can define D^2-finite sequences as sequences satisfying a linear recurrence with D-finite coefficients.
- Example: Superfactorial $a(n) = \prod_{k=1}^{n} k!$ (A000178).
- Let us define C^k-finite (or D^k-finite) sequences as sequences satisfying a linear recurrence with C^{k-1}-finite (or D^{k-1}-finite) coefficients.
- Using the same methods as for C^2-finite sequences: All these sets form rings (Jiménez-Pastor, N., and Pillwein 2022).
- Let c be C-finite. Then, $c(n^k)$ is C^k-finite.

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[2] Antonio Jiménez-Pastor, P. N., and Veronika Pillwein. "An extension of holonomic sequences: C^2-finite sequences". In: Journal of Symbolic Computation (2022). `https://doi.org/10.35011/risc.21-20`.

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[5] Tomer Kotek and Johann A. Makowsky. “Recurrence relations for graph polynomials on bi-iterative families of graphs”. In: Eur. J. Comb. 41 (2014), pp. 47–67.

[6] P. N. and Veronika Pillwein. “A comparison of algorithms for proving positivity of linearly recurrent sequences”. In: Proceedings of CASC 2022. 2022.

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[8] Joël Ouaknine and James Worrell. “Decision Problems for Linear Recurrence Sequences”. In: Lecture Notes in Computer Science. Springer, 2012, pp. 21–28.