C-FINITE AND C^2-FINITE SEQUENCES IN SAGEMATH

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rec_sequences package

- rec_sequences is a SageMath package for linear recurrence sequences (mostly C-finite and C^2-finite sequences).
- Based on the ore_algebra package (Kauers, Jaroschek, and Johansson 2015).
- Can be obtained from GitHub: github.com/PhilippNuspl/rec_sequences

```
sage: from rec_sequences.CFiniteSequenceRing import *
sage: from rec_sequences.C2FiniteSequenceRing import *
```

C-finite sequences

Let $\mathbb{K} \supseteq \mathbb{Q}$ be a number field.

Definition

A sequence $c(n) \in \mathbb{K}^{\mathbb{N}}$ is called C-finite if there are constants $\gamma_0, \ldots, \gamma_r \in \mathbb{K}$, not all zero, such that

$$\gamma_0 c(n) + \cdots + \gamma_r c(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

```
sage: C = CFiniteSequenceRing(QQ)
sage: fib = C([1,1,-1], [1,1]) # use recurrence and initial values
sage: fib[:10]
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
sage: alt = C(10*[1 , -1]) # use initial values and guessing
sage: alt
C-finite sequence a(n): (1)*a(n) + (1)*a(n+1) = 0 and a(0)=1
```

C-finite identities and inequalities

- Using closure properties of C-finite sequences (implemented in the ore_algebra package) one can prove and derive identities.

```
sage: fib.sum() == fib.shift(2) - 1
True
sage: cassini = fib^2 - fib.shift()*fib.shift(-1)
sage: cassini.closed_form() # Cassini identity
-(-1)^n
```

- Several methods for proving C-finite (termwise) inequalities are implemented:

```
sage: var("n");
sage: exp2 = C(2^n)
sage: fib <= exp2, 10 > fib, alt >= -1
(True, False, True)
sage: alt.sign_pattern()
Sign pattern: cycle <+->
```

C^2-finite sequences

Definition

A sequence $a = a(n) \in \mathbb{K}^{\mathbb{N}}$ is called C^2-finite if there are C-finite sequences $c_0(n), \ldots, c_r(n) \in \mathbb{K}^{\mathbb{N}}$ with $c_r(n) \neq 0$ for all $n \in \mathbb{N}$ such that

$$c_0(n)a(n) + \cdots + c_{r-1}(n)a(n+r-1) + c_r(n)a(n+r) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Example: Fibonorials $a(n) = \prod_{k=0}^{n-1} f(k)$ where $f(n)$ denotes the Fibonacci sequence. They satisfy

$$f(n)a(n) - a(n+1) = 0 \text{ for all } n \in \mathbb{N}.$$

```
sage: C2 = C2FiniteSequenceRing(QQ)
sage: fib_fac = C2([fib,-1], [1])
sage: fib_fac[:10] # fibonacci-factorial, A003266
[1, 1, 1, 2, 6, 30, 240, 3120, 65520, 2227680]
```

Example: Sparse Subsequences

- For a C-finite sequence $c(n)$, the sequence $c(n^2)$ is C^2-finite.
- In particular $f(n^2)$ is C^2-finite satisfying (Kotek and Makowsky 2014)

$$f(2n+3)f(n^2) + f(4n+4)f((n+1)^2) - f(2n+1)f((n+2)^2) = 0.$$

```
sage: fib_sparse = fib.sparse_subsequence(C2) # A054783
sage: fib_sparse
C^2-finite sequence of order 2 and degree 2 with coefficients:
   > c0 (n) : C-finite sequence c0(n): (-1)*c0(n) + (3)*c0(n+1) +
    (-1)*c0(n+2) = 0 and c0(0)=-2 , c0(1)=-5
   > c1 (n) : C-finite sequence c1(n): (-1)*c1(n) + (7)*c1(n+1) +
    (-1)*c1(n+2) = 0 and c1(0)=-3 , c1(1)=-21
   > c2 (n) : C-finite sequence c2(n): (-1)*c2(n) + (3)*c2(n+1) +
    (-1)*c2(n+2) = 0 and c2(0)=1 , c2(1)=2
and initial values a(0)=1 , a(1)=1
sage: fib_sparse[:10]
[1, 1, 5, 55, 1597, 121393, 24157817, 12586269025]
```

Ring (closure properties)

Theorem (Jiménez-Pastor, N., and Pillwein 2021)

The set of C^2-finite sequences is a difference ring with termwise addition and termwise multiplication.

```
sage: a = C2([alt, exp2, -1], [1, 1])
sage: b = C2([1, exp2, alt], [1, 1])
sage: g = a+b
sage: 0 not in g.leading_coefficient()
True
sage: g.order(), g.degree()
(4, 4)
sage: g[:100] == [an+bn for an, bn in zip(a[:100], b[:100])]
True
```

More closure properties

C^2-finite sequences are also closed under

- partial sums,
- taking subsequences at arithmetic progressions,
- interlacing.

Example

The sequence $\sum_{k=0}^{\lfloor n/3 \rfloor} f((2k+1)^2)$ is C^2-finite.

```
sage: a = fib_sparse.subsequence(2, 1).sum().multiple(3)
sage: a.order(), a.degree()
(9, 147)
sage: a2 = fib.sparse_subsequence(C2, 4, 4, 1).sum().multiple(3)
sage: a2.order(), a2.degree()
(9, 12)
```

Guessing

- Know that $f(n^2)$ also satisfies a simple C^2-finite recurrence (i.e., with leading coefficient 1).
- Can use guessing to find such a recurrence. Make ansatz

$$c_0(n)f(n^2) + c_1(n)f((n+1)^2) + c_2(n)f((n+2)^2) + f((n+3)^2) = 0.$$

Assume $c_i(n) = \sum_{j=0}^{m} \gamma_{i,j}\lambda_j^n$ for particular λ.

```
sage: K.<u> = NumberField(x^2-5)
sage: Cext = CFiniteSequenceRing(K)
sage: C2ext = C2FiniteSequenceRing(K)
sage: phi, psi = (1+u)/2, (1-u)/2
sage: zeros = [phi^4, psi^4, phi^6, psi^6, phi^8, psi^8]
sage: data = [fib[n^2] for n in range(100)]
sage: sparse_fib_simple = C2ext.guess(data, zeros, 3, simple=True)
sage: sparse_fib_simple.leading_coefficient() == 1
True
```

References

Jiménez-Pastor, Antonio, Philipp N., and Veronika Pillwein (2021). "On C^2-finite sequences". In: Proceedings of ISSAC 2021, Virtual Event Russian Federation, July 18–23, 2021, pp. 217–224.

Kauers, Manuel, Maximilian Jaroschek, and Fredrik Johansson (2015). "Ore Polynomials in Sage". In: Computer Algebra and Polynomials: Applications of Algebra and Number Theory. Springer, pp. 105–125.

Kotek, Tomer and Johann A. Makowsky (2014). "Recurrence relations for graph polynomials on bi-iterative families of graphs". In: Eur. J. Comb. 41, pp. 47–67.