Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

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Reminder: Do not forget to mark crosses online until Oct.10, 9 am. at https://fox.indmath.uni-linz.ac.at/wwwlehre/index.php?course=intgl

1. Let ℓ_∞ be the linear space of bounded sequences with the norm $\|x\|_\infty := \sup_{n=1}^\infty |x_n|$. Let $A : \ell_\infty \to \ell_\infty$ be defined by

$$A(x) = A(x_1, x_2, \dots) := (x_1, x_2/2, x_3/3, \dots)\,.$$

Show that A is injective, continuous and not surjective. Furthermore, show that $\|A\| = 1$ and that A^{-1} is not continuous.

2. Let $K \in \mathbb{R}^{m\times n}$, $f \in \mathbb{R}^m$. Prove the following statement: $Kx = f$ has a solution in $\mathbb{R}^n$ if and only if all solutions $y \in \mathbb{R}^m$ of $K^T y = 0$ satisfy $(y, f)_{\mathbb{R}^m} = 0$.

3. Classify the following integral equations and identify, if possible, the kernel in each equation:

   (a) $\alpha f(x) = \displaystyle\int_{-t}^{t} (x-y)^2 \sqrt{f(y)}\, dy + s(x) f(x)\,,$

   (b) $\displaystyle\int_0^s (1+\beta) x(t)\, dt = (\beta - e^{i\pi}) x(s)\,,$

   (c) $\sin(2\pi s) - \displaystyle\int_u^v 2\cos(\pi x) f(x)\, dx = g(s)\,.$

4. Rewrite the following boundary value problem using an integral equation and classify it:

$$y''(s) + \lambda y(s) = 0,\ y(0) = y(1) = 0,\ s \in [0,1]\,.$$

5. Let $A : X \to X$ be a mapping on a Banach space X, with the property that A^n is contractive on X for some n. In other words,

$$||A^n x - A^n y\| \le q\|x - y\|\,,$$

holds for all $x, y \in X$, $q < 1$, and some $n \in \mathbb{N}$. Please show: A has exactly one fixed point.

6. We consider an integral equation on the interval $[a,b]$:

$$x(s) - \int\limits_a^s k(s,t)x(t)\,dt = g(s)\,, \qquad s \in [a,b]\,,$$

where $g : [a,b] \to \mathbb{R}$ and $k : \{(s,t) \in \mathbb{R}^2 : a \le t \le s \le b\} \to \mathbb{R}$ are continuous functions. Please show: The integral equation above has exactly one solution.

Hint: Let $A : C[a,b] \to C[a,b]$ be given by $(Ax)(s) = \int\limits_a^s k(s,t)x(t)\,dt + g(s)$ for $s \in [a,b]$. Show and use:

$$|(A^n x)(s) - (A^n y)(s)| \le c^n \frac{(s-a)^n}{n!} \|x-y\|_\infty\,,$$

holds for all $x, y \in C[a,b]$, $s \in [a,b]$, and $n \in \mathbb{N}$, with $c := \max_{a \le t \le s \le b} |k(s,t)|$.