Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

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Dr. Simon Hubmer, S2 503-1, simon.hubmer@ricam.oeaw.ac.at

55. Let $\langle X, Y \rangle$ be a dual system with two normed spaces, X and Y. Let $A_n : X \to X$ and $B_n : Y \to Y$ be adjoint finite dimensional operators of the form,

$$A_n \phi = \sum_{j=1}^{n} \langle \phi, b_j \rangle \, a_j \quad \text{and} \quad B_n \psi = \sum_{j=1}^{n} \langle a_j, \psi \rangle \, b_j \,,$$

for $\phi \in X$ and $\psi \in Y$, and linearly independent elements $\{a_1, \dots, a_n\} \subset X$ and $\{b_1, \dots, b_n\} \subset Y$. By reducing the operator equations to linear systems, demonstrate the validity of Fredholm's theorem for the operators $I - A_n$ and $I - B_n$.

56. Let $\langle X, Y \rangle$, A_n and B_n be defined as in the previous problem. Let $S : X \to X$ and $T : Y \to Y$ be adjoint linear operators such that $\|S\| < 1$ and $\|T\| < 1$. Establish the validity of Fredholm's theorem for the operators $I - A$ and $I - B$ where

$$A = A_n + S \quad \text{and} \quad B = B_n + T \,.$$

Hint: Establish that $(I - S)$ and $(I - T)$ have bounded inverse operators on X and Y, respectively. Then transform the operator equations into equivalent equations with the operators $I - (I - S)^{-1} A_n$ and $I - (I - T)^{-1} B_n$.

57. Let K be a compact operator induced by the kernel $k(s,t) \in L^2([0,1]^2)$ with $k(s,t) > 0$. Prove that $\|K\| < 1$ if and only if $(I - K)$ has a bounded inverse $(I - K)^{-1}$ which is induced by a positive kernel.

58. From the proof of Theorem 2.1, it was shown that for $k \in C(G \times G)$ and $x \in L^2(G)$ the operator K induced by k satisfies the inequality,

$$\|Kx\|_2 \leq \|k\|_2 \, \|x\| \;,$$

i.e., $\|K\|_2 \leq \|k\|_2$. Is this actually the norm? In other words, does it hold that

$$\|K\|_2 = \|k\|_2 \,?$$

Prove or show a counterexample.
Hint: Consider the finite dimensional case (i.e., K is a matrix).

59. Please solve:

$$\int_0^s (s+1-t)x(t)\,dt = e^s - 1\,, \qquad s \in [0,1]\,.$$

for $x \in C[0,1]$.

Hint: use Theorem 4.3.

60. Please show that the following integral equation

$$x(s) - \int_0^1 k(s,t)x(t)\,dt = s\,,$$

where the kernel k is given by

$$k(s,t) = \begin{cases} 1+t\,, & t \leq s\,, \\ 2s\,, & t > s\,, \end{cases}$$

has no solutions in $L^2[0,1]$.

Hint: Show and use: The integral equation

$$x(s) = \int_0^s 2tx(t)\,dt + \int_s^1 (1+s)x(t)\,dt\,,$$

can be reformulated as a differential equation

$$\begin{cases} x''(s) & = (s-1)x'(s)\,, \\ x'(0) & = x'(1) = 0\,, \end{cases}$$

for $x \in C^2[0,1]$.