Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

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Dr. Simon Hubmer, S2 503-1, simon.hubmer@ricam.oeaw.ac.at

67. Calculate the Green's Function for the following boundary value problem

$$-x''(s) + x(s) = f(s)\,, \quad s \in [0,1]\,, \quad x(0) = 0\,, \quad x'(1) = 0\,.$$

68. Define $(Lx)(s) := -x''(s) + x(s)$, the operator from Exercise 67. Let L_B be defined as the restriction of the operator L o the space of functions satisfying the boundary conditions. Calculate the eigenvalues and eigenfunctions of L_B.

69. Calculate the Green's Function for the boundary value problem

$$x''(s) = f(s)\,, \quad s \in [0,1]\,, \quad x(0) = x'(0)\,, \quad x(1) = -x'(1)\,.$$

70. Let

$$D_B := \{x \in C^2[0,1] \,|\, x(0) - x(1) = 0\,, x'(0) - x'(1) = 0\}\,,$$

and define $Lx := -x''$ and $L_B := L|_{D_B}$.

(a) Calculate the eigenvalues and eigenfunctions of L_B.

(b) Why in (58a) does no contradiction arise to Theorem 5.8, part (d), that all eigenspaces associated to eigenvalues of a Sturm Liouville Problem are one dimensional? Note: The requirement that 0 is not an eigenvalue was not used in the proof of part (d).

71. Let $K : L^2[0,1] \to L^2[0,1]$ be induced by the kernel $k(s,t) = \min(s,t)$, i.e.,

$$(Kx)(s) = \int_0^1 \min(s,t) x(t) dt\,.$$

- Show that (λ, x), with $\lambda \in \mathbb{R}$ and $x \in L^2[0,1]$, is an eigenpair of K if and only if the (λ, x) satisfies

$$\lambda x''(s) + x(s) = 0 \ \text{ for } s \in (0,1)\,,$$

with boundary conditions $x(0) = x'(1) = 0$.

- Deduce that $\lambda_i = \frac{1}{(n-1/2)^2\pi^2}$ and $x_n(s) = \sqrt{2}\sin((n-1/2)\pi s)$ are the eigenvalues and eigenvectors, respectively.

72. Let $x \in C^2[a,b]$ and let $g(s,t)$ be a Green's Function including the integral operator G, as described in Theorem 5.5. In the proof of Theorem 5.5, we have

$$x(s) := \int_a^s g(s,t) f(t)\, dt + \int_s^b g(s,t) f(t)\, dt\,.$$

It follows that

$$x'(s) = \int_a^s \frac{\partial g}{\partial s}(s,t) f(t)\, dt + \int_s^b \frac{\partial g}{\partial s}(s,t) f(t)\, dt\,, \tag{1}$$

and then

$$x''(s) = \frac{f(s)}{p(s)} + \int_a^b \frac{\partial^2 g}{\partial^2 s}(s,t) f(s)\, ds\,. \tag{2}$$

Please fill in the details for the proof showing that (2) follows from (1), specifically as it relates to the fact that $\frac{\partial g}{\partial s}(s,t)$ is discontinuous at (s,s).