Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

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7. Let the functions f, p, q be fixed and continuous in the closed interval $a \leq s \leq b$ and $c_0, c_1 \in \mathbb{R}$ such that p is continuously differentiable and has no roots on the interval $[a, b]$, i.e., $p(s) \neq 0, \forall s \in [a, b]$. Rewrite the following initial value problem using an integral equation and classify it:

$$(py')'(s) + q(s)y(s) = f(s), \; y(a) = c_0, \; y'(a) = c_1 \,.$$

8. Does the Fredholm integral equation

$$x(s) - \int_0^1 x(t)\, dt = s \,,$$

have solutions in $C[0, 1]$? Why/why not?

9. Please show that the integral equation

$$g(s) = f(s) + \frac{1}{\pi} \int_0^{2\pi} \sin(s + t) g(t)\, dt \,,$$

has no solution for $f(s) = s$. What can be said about the case $f(s) = 1$? *Hint:* Show and use that the kernel is degenerate.

10. Let $\{\phi_1, \phi_2, \ldots, \phi_n, \ldots\}$ be an orthonormal basis of $L^2(G)$ with $G \subset \mathbb{R}^n$. Let $\alpha_{i,j}(s, t) := \phi_i(s)\phi_j(t)$. Show that $\{\alpha_{i,j}\}_{i,j\in\mathbb{N}}$ is a complete orthonormal system of $L^2(G \times G)$.

11. Solve the following integral equation:

$$g(s) - \lambda \int_0^1 (20st^2 + 12s^2t) g(t)\, dt = s \,, \qquad s \in [0, 1] \,.$$

12. Let the kernel k be given by

$$k(s,t) := \begin{cases} s(1-t) & s \leq t \,, \\ t(1-s) & s > t \,, \end{cases} \qquad s, t \in [0, 1] \,,$$

and the integral operator $K : C[0,1] \to C[0,1]$ be given by

$$(Kx)(s) := \int\limits_0^1 k(s,t)x(t)\,dt\,.$$

Show that the range of K is a subset of $D := \{f \in C^2[0,1], f(0) = f(1) = 0\}$. Show that, indeed, $\mathcal{R}(K) = D$. In other words, show that $f \in \mathcal{R}(K)$ if and only if $f \in D$.
Hint: **To show** $f \in \mathcal{R}(K) \Rightarrow f \in D$, notice that all elements of $\mathcal{R}(K)$ are given by $\int_0^1 k(s,t)x(t)\,dt$ with $x \in C[0,1]$, i.e., it is enough show that $\int_0^1 k(s,t)x(t)\,dt \in D$. **To show** $f \in D \Rightarrow f \in \mathcal{R}(K)$ K, observe that by definition, $f \in \mathcal{R}(K) \iff$ there exists $x \in C[0,1] : Kx = f$. Let $f \in D$. Try differentiating both sides to find an $x \in C[0,1]$ which solves $Kx = f$.