# Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

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19. Let the assumptions of Exercise 5 hold. Please show that for some initial value $x_0 \in X$, the sequence of *successive approximations* given by $x_{k+1} := Ax_k$, $k \in \mathbb{N}$ converges to the fixed point of $A$.

20. Let $X, Y$ be normed vector spaces. Let $K : X \to Y$ be a linear operator and let $X$ be finite dimensional. Please show: $K$ is compact.

21. We consider the following integral equation

$$x(s) - \int_0^s x(t)\,dt = 1\,, \qquad s \in [0,1]\,.$$

The exact solution is $x(s) = e^s$. We approximate the Volterra kernel

$$k(s,t) := \begin{cases} 1, & \text{if } t \leq s\,, \\ 0, & \text{if } t > s\,, \end{cases}$$

with degenerate kernels

$$k_n(s,t) := \sum_{i=1}^{n} \chi_{(\frac{i-1}{n},\frac{i}{n}]}(s) \cdot \chi_{[0,\frac{i-1}{n}]}(t)\,,$$

where $\chi_I$ are the characteristic functions on the interval $I$. Compute the solutions $x_n$, $n \in \mathbb{N}$ of

$$x_n(s) - \int_0^1 k_n(s,t) x_n(t)\,dt = 1\,, \qquad s \in [0,1]\,,$$

and the limit as $n \to \infty$, if it exists.

*Hint:* Notice that $x_n$ is piecewise constant on the intervals $((i-1)/n, i/n]$, $i = 1, \ldots, n$. Its values

$$x_n(s) = 1 + \int_0^1 k_n(s,t) x_n(t)\,dt, \qquad s \in \left(\frac{i-1}{n}, \frac{i}{n}\right]\,,$$

can be computed recursively for $i = 1, \ldots, n$.

22. Let the integral operator $K : C[0,1] \to C[0,1]$ be induced by the kernel

$$k(s,t) := \frac{|s-t|^{1/4}}{\sin|s-t|}\,, \qquad s,t \in [0,1]\,, s \neq t\,.$$

Please show: $K$ is compact.

*Hint:* Notice that for $u \in (0,1]$:

$$\frac{u^{1/4}}{\sin u} = \frac{u}{\sin u} \cdot \frac{1}{u^{3/4}}\,.$$

23. Show that the integral equation

$$\phi(x) = \sin(x) + \lambda \int_0^{\pi/2} \cos(x-y)\phi(y)\,dy\,,$$

has a solution for $\lambda \neq \frac{4}{\pm 2+\pi}$ (and calculate it).

24. Solve the following ordinary differential equation

$$\begin{cases} x''(t) + 2x'(t) = -8\sin(2t)\,, \\ x(0) = 1, x'(0) = 2\,, \end{cases}$$

using the Laplace transform.
*Hint:* You can refer to tables for some known Laplace transforms.