Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

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25. Show that the integral operator $K : C[0,1] \to C[0,1]$ induced by the singular kernel

$$k(s,t) = \log|s-t|\,, \qquad s,t \in [0,1], s \neq t\,,$$

is compact.

26. Let X be a Banach space and $K : X \to X$ a bounded, linear operator with $\|K\| < 1$. Show that in this case $(I-K)^{-1}$ exists and is continuous.

27. Let X, Y be Hilbert spaces and $A : X \to Y$ linear and continuous. Show that for A and its adjoint $A^* : Y \to X$ it holds:

$$\mathcal{N}(A^*) = \mathcal{R}(A)^\perp, \qquad \text{and} \qquad \overline{\mathcal{R}(A^*)} = \mathcal{N}(A)^\perp\,.$$

28. Let X be a linear space and let $A, B : X \to X$ be linear operators with $AB = BA$. Let the operator AB have an inverse $(AB)^{-1} : X \to X$. Show that A and B have the respective inverse operators

$$A^{-1} = B(AB)^{-1} \text{ and } B^{-1} = A(AB)^{-1}\,.$$

29. Show, that

$$x(s) - \frac{1}{2}\int_0^1 \cos(st)x(t)\,dt = f(s), \qquad s \in [0,1]\,,$$

has a unique solution $x \in C[0,1]$ for all $f \in C[0,1]$.

Hint: Let $(Kx)(s) := \frac{1}{2}\int_0^1 \cos(st)x(t)\,dt$. Show and use that for $x \in \mathcal{N}(I-K)$

$$|x(s)| \leq \frac{1}{2}\|x\|_\infty\,, \qquad s \in [0,1]\,,$$

holds and implies that $\mathcal{N}(I-K) = \{0\}$.

30. Let $k(s,t) := \chi_{[0,1]}(s)\chi_{[0,1]}(t) + \chi_{[0,2]}(s)\chi_{[1,2]}(t)$ for $s,t \in [0,2]$ and let $K : L_2[0,2] \to L_2[0,2]$ be an integral operator induced by k. Please show that the Riesz index of $I-K$ is $\nu = 2$.