Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

Exercise sheet 6 — 14.11.2018

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31. Find a normed space X and a compact operator $K : X \to X$ for which
$$X \neq \mathcal{N}(I-K) \oplus \mathcal{R}(I-K)\,.$$

32. Let X, Y, Z be normed spaces, $B : X \to Y$ and $A : Y \to Z$ be linear and bounded operators. Show or find a counter-example: the operator $C = AB$ is compact if and only if A or B are compact.

33. Let X be a Banach space and let X^* be its dual space. Let $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$ be the duality mapping given by $\langle f, x\rangle := f(x)$. Please show that $\langle\cdot,\cdot\rangle$ is a non-degenerate bilinear form.

    Hint: To show $\forall (x \in X, x \neq 0)$ there exists $f \in X^* : f(x) \neq 0$, first construct a linear bounded operator $f : Y_x \to \mathbb{R}$, where $Y_x = \{z \in X : z = \alpha x, \alpha \in \mathbb{R}\}$ for a fixed $x \in X$. As Y_x is a linear subspace of X, use the Hahn-Banach Theorem to argue the existence of a linear bounded f defined on the whole space X.

34. Let $(X, Y, (\cdot,\cdot))$ be a dual system. Please show that if $A : X \to X$ has an adjoint operator $B : Y \to Y$, then B is unique and both A and B are linear.

35. Let the normed space X be given by
$$X := \{x \in C(0,1] : \exists M, \alpha > 0 : |x(t)| \leq M t^{\alpha - \frac{1}{2}}, t \in (0,1]\}\,,$$
with the norm
$$\|x\| := \sup_{t\in(0,1]} \sqrt{t}|x(t)|\,.$$
Please show that
$$(x,y) := \int_0^1 x(t)y(t)\,dt\,, \qquad x, y \in X\,,$$
is a non-degenerate unbounded bilinear form on $X \times X$.

    Hint: To show unboundedness, consider the choice $x = y = x_n := t^{1/n - 1/2}$.

36. Please show that for $B \in \mathbb{R}^{n\times n}$ holds
$$\nu \text{ is Riesz index of } B \iff \nu = \inf\{s \in \mathbb{N}_0 \mid \text{rank } B^s = \text{rank } B^{s+1}\}\,.$$