Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

Exercise sheet 7 21.11.2018

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37. Show that a symmetric matrix has Riesz index 0 or 1. Furthermore, give an example for a non symmetric matrix with Riesz index 0 or 1.

38. Let H be a complex Hilbert space with inner product $(\cdot,\cdot)$. Furthermore, let $A : X \to X$ be linear, bounded, self adjoint and positiv semidefinit. Show that it holds
$$|(Ax, y)|^2 \le (Ax, x)(Ay, y)\,, \qquad x, y \in H\,.$$

39. Let K be the integral operator induced by the kernel $k(s,t) := e^{-t^2(s+1)}$ on $C[0,1]$ with the supremum-norm. Calculate the Riesz index of $L := I - K$.

40. Consider the integral equation
$$x(s) - \int_0^1 k(s,t)x(t)\,dt = f_i(s)\,, \quad s \in [0,1]\,,\ i = 1,2,3\,,$$
with $k(s,t)$ from Exercise 39 and $f_1(x) = \sin s$, $f_2(x) = s^3 - e^s$ and $f_3(s) = e^{s^2}$. For which of the right hand sides is the integral equation solvable?

41. Let $A : X \longrightarrow X$ be a bounded linear operator mapping a Banach space X to itself with $\|A^k\| < 1$ for some $k \in \mathbb{N}$. Let $I : X \longrightarrow X$ denote the identity operator on X. Show that
$$\phi_{n+1} = A\phi_n + f,\ \ n = 0, 1, 2, \dots\,,$$
with some arbitrary $\phi_0 \in X$ converges to the unique solution of
$$\phi - A\phi = f\,.$$

42. Let $(\cdot,\cdot) : C[a,b] \times C[a,b] \to \mathbb{R}$ be a degenerate bilinear form. In other words there exists an $x \in C[a,b], x \neq 0$, such that for all $y \in C[a,b]$, $(x,y) = 0$. Without loss of generality, we may assume that $x(a) = 1$. Consider the operators A and B given by
$$\begin{aligned} A : C[a,b] &\to C[a,b]\,, & B : C[a,b] &\to C[a,b]\,,\\ \phi &\mapsto \phi(a)x\,, & \psi &\mapsto 0\,. \end{aligned}$$
Please show that A and B are compact and adjoint w.r.t. $(\cdot,\cdot)$.
Hint: Show and use: A is bounded and $\dim \mathcal{R}(A) < \infty$ to conclude that A is compact.