Integral Equations and Boundary Value Problems

Exercise, WS 2018/19

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For Exercises 43 and 44 assume the following. Let $(X, Y, (\cdot,\cdot))$ be a dual system. Let $S : X \to X$ and $T : Y \to Y$ be linear continuously invertible operators on X and Y respectively and let $A : X \to X$ and $B : Y \to Y$ be compact operators, such that S is adjoint to T and A is adjoint to B.

43. Under the assumptions above please show that: the homogeneous equations
$$Sx - Ax = 0\,,$$
and
$$Ty - By = 0\,,$$
have the same number of linearly independent solutions.
Hint: You may use the Fredholm's alternative. Use and show that S^{-1} is adjoint to T^{-1}, $S^{-1}A$ is adjoint to BT^{-1}, $\mathcal{N}(S - A) = \mathcal{N}(I - S^{-1}A)$ and $\dim \mathcal{N}(T - B) = \dim \mathcal{N}(I - BT^{-1})$.

44. Under the assumptions above please show that: the inhomogeneous equation
$$Sx - Ax = f, \qquad f \in X\,,$$
has a solution if and only if for all solutions y of $Ty - By = 0$, $(f, y) = 0$.
Hint: You may use the Fredholm's alternative.

45. Compute the nullspaces $\mathcal{N}(I - A)$ and $\mathcal{N}(I - B)$ for A and B from Exercise 42.

46. Let $a \in C[0,1]$. The bounded linear operator $A : C[0,1] \to C[0,1]$ is given by $(Ax)(s) = a(s)x(s)$ for $s \in [0,1]$ and $x \in C[0,1]$. Please show:
The spectrum of A is given by $\sigma(A) = \{a(s) : s \in [0,1]\}$.
To (a): try to invert $(I - A)$.

47. Let A be as in Exercise 46. Please show:
   - If A is compact, then $A = 0$.
   - A has an eigenvalue, if and only if a is constant on an interval.

Hint:

To (a): show that if A is compact, then $0 \in \sigma(A)$; furthermore, $\sigma(A)$ must be at most a countable set.
To (c), "$\Longleftarrow$": find an $x \in C[0,1], x \neq 0$ such that $x \in \mathcal{N}(\lambda I - A)$.

48. Compute the spectrum of $K : C[0,1] \to C[0,1]$ given by

$$(Kx)(s) := \int_0^s x(t)\,dt\,, \qquad s \in [0,1]\,.$$

Hint: Show that $0 \in \sigma(K)$. Show that for all $\lambda \neq 0$, $\lambda I - K$ is continuously invertible on $C[0,1]$ or that λ is not an eigenvalue of K.