On the Inverse Problem of Linearized Elasticity

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Introduction and Motivation

Elastography is an imaging modality that can distinguish materials by their biomechanical properties [1]. Palpation exams at the doctors for detecting abnormal tissue motivates the development of quantitative elasticity imaging. A specimen of study is subjected to a deformation and the resulting internal displacement field is measured, which provides data for identifying the Lamé parameters. In short, we face the following

Problem: Identify the Lamé parameters from displacement measurements.

Mathematical Model

Given a bounded, open and connected set $\Omega \in \mathbb{R}^N$, $N = 2,3$, with Lipschitz continuous boundary $\partial\Omega = \overline{\Gamma_D \cup \Gamma_T}$, $\Gamma_D \cap \Gamma_T = \emptyset$, meas$(\Gamma_D) > 0$ and body forces f, prescribed displacement g_D, surface traction g_T and **Lamé parameters** λ and μ, the homogenized equations of **linearized elasticity** with displacement-traction boundary conditions are given by

$$\begin{aligned} -\operatorname{div}(\sigma(u)) &= f + \operatorname{div}(\sigma(\Phi)), \quad \text{in}\,\Omega, \\ u|_{\Gamma_D} &= 0, \\ \sigma(u)\vec{n}|_{\Gamma_T} &= g_T - \sigma(\Phi)\vec{n}|_{\Gamma_T}, \end{aligned} \tag{1}$$

where $\vec{n}$ is an outward unit normal, Φ is a function such that $\Phi|_{\Gamma_D} = g_D$, **the strain tensor** $\mathcal{E}$ and **the stress tensor** σ defining the stress-strain relation in Ω are given by

$$\mathcal{E}(u) := \frac{1}{2}\left(\nabla u + \nabla u^T\right), \qquad \sigma(u) := 2\mu\,\mathcal{E}(u) + \lambda\,\operatorname{div}(u)\,I. \tag{2}$$

Inverse Problem

We want to precisely define our inverse problem. For this, we first introduce

$$V := H^1_{0,\Gamma_D}(\Omega)^N := \{u \in H^1(\Omega)^N \,|\, u|_{\Gamma_D} = 0\}, \tag{3}$$

as well as the linear and the bilinear forms

$$l(v) := \langle f, v\rangle_{H^{-1}(\Omega),H^1(\Omega)} + \langle g_T, v\rangle_{H^{-\frac{1}{2}}(\Gamma_T),H^{\frac{1}{2}}(\Gamma_T)}, \tag{4}$$

$$a_{\lambda,\mu}(u,v) := \int_\Omega \left(\lambda\,\operatorname{div}(u)\operatorname{div}(v) + 2\mu\,\mathcal{E}(u) : \mathcal{E}(v)\right) dx. \tag{5}$$

Using this, the linearized elasticity problem (1) can be written in the weak form

$$a_{\lambda,\mu}(u,v) = l(v) - a_{\lambda,\mu}(\Phi, v), \qquad \forall\, v \in V. \tag{6}$$

Introducing the nonlinear operator called **parameter-to-solution map**

$$\begin{aligned} F : \mathcal{D}(F) := \left\{(\lambda,\mu) \in L^\infty(\Omega)^2 \,|\, \lambda \geq 0,\, \mu \geq \underline{\mu} > 0\right\} \subset L^\infty(\Omega)^2 &\to L^2(\Omega)^N, \\ (\lambda,\mu) &\mapsto u(\lambda,\mu), \end{aligned} \tag{7}$$

where $u(\lambda,\mu)$ is defined as the solution of (6), our problem now reads as follows:

Problem. Given $f \in H^{-1}(\Omega)^N$, $g_D \in H^{\frac{1}{2}}(\Gamma_D)^N$, $\Phi \in H^1(\Omega)^N$, $g_T \in H^{-\frac{1}{2}}(\Gamma_T)^N$ and a measurement $u^\delta \in L^2(\Omega)^N$ of the true displacement field $u \in V$ satisfying $\|u - u^\delta\|_{L^2(\Omega)} \leq \delta$, compute an approximation of the **Lamé parameters** λ and μ, which satisfy

$$F(\lambda,\mu) = u. \tag{8}$$

Derivative and Adjoint

Defining the operator $\tilde{A}_{\lambda,\mu}$ connected to the bilinear form $a_{\lambda,\mu}$ by

$$\tilde{A}_{\lambda,\mu} : H^1(\Omega)^N \to V^*, \qquad u \mapsto \left(v \mapsto a_{\lambda,\mu}(u,v)\right), \tag{9}$$

and its restriction to V, i.e., $A := \tilde{A}|_V$, the operator F can be written in the alternative form:

$$F(\lambda,\mu) = A^{-1}_{\lambda,\mu}\left(l - \tilde{A}_{\lambda,\mu}\Phi\right). \tag{10}$$

Theorem 1. *F is a well-defined, continuously Fréchet differentiable operator satisfying*

$$F'(\lambda,\mu)(h_\lambda,h_\mu) = -A^{-1}_{\lambda,\mu}\left(A_{h_\lambda,h_\mu}u(\lambda,\mu) + \tilde{A}_{h_\lambda,h_\mu}\Phi\right). \tag{11}$$

Since F is defined on the non-reflexive Banach space $L^\infty(\Omega)^2$, our problem does not fit into the standard Banach or Hilbert space theory [2, 3]. Hence, we embed the problem into a suitable Hilbert space by defining the following restriction of F:

$$\begin{aligned} \tilde{F} : \{(\lambda,\mu) \in H^2(\Omega)^2 \,|\, \lambda \geq 0, \mu \geq \underline{\mu} > 0\} \subset H^2(\Omega)^2 &\to L^2(\Omega)^N, \\ (\lambda,\mu) &\mapsto \tilde{F}(\lambda,\mu) := F(\lambda,\mu), \end{aligned} \tag{12}$$

and instead of (8) restrict ourselves to the problem $\tilde{F}(\lambda,\mu) = u$, which is much easier to treat. $\tilde{F}$ is also continuously Fréchet differentiable and furthermore, we have the following

Theorem 2. *The adjoint of the Fréchet derivative of $\tilde{F}$ is given by*

$$\tilde{F}'(\lambda,\mu)^* w = \begin{pmatrix} E\left(\operatorname{div}\left(u(\lambda,\mu) + \Phi\right)\operatorname{div}\left(-A^{-1}_{\lambda,\mu}Tw\right)\right) \\ E\left(2\mathcal{E}\left(u(\lambda,\mu) + \Phi\right) : \mathcal{E}\left(-A^{-1}_{\lambda,\mu}Tw\right)\right) \end{pmatrix}^T, \tag{13}$$

where T and E are defined by

$$T : L^2(\Omega)^N \to V^*,\; w \mapsto \left(v \mapsto \int_\Omega w \cdot v\,dx\right), \qquad E : L^1(\Omega) \to H^2(\Omega),\; \langle Eu, v\rangle_{H^2(\Omega)} = \int_\Omega uv\,dx.$$

Reconstruction Algorithm

In order to solve (8) we use different **Landweber type gradient methods**, i.e., together with the abbreviation $x_k^\delta = (\lambda_k^\delta, \mu_k^\delta)$, we employ the following family of gradient methods:

$$x_{k+1}^\delta = x_k^\delta + \omega_k^\delta(x_k^\delta)\, s_k^\delta(x_k^\delta), \qquad s_k^\delta(x) := F'(x)^*\left(u^\delta - F(x)\right), \tag{14}$$

where for the stepsize ω_k^δ we use both the **steepest descent** stepsize

$$\omega_k^\delta(x) := \frac{\|s_k^\delta(x)\|^2}{\|F'(x)s_k^\delta(x)\|^2}, \tag{15}$$

and the recently introduced [4] stepsize

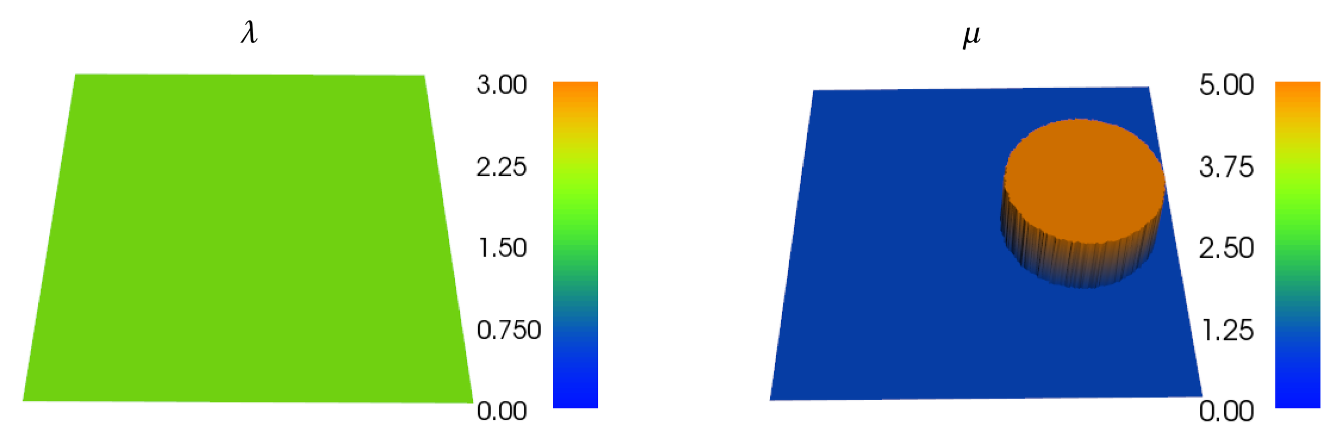
$$\omega_k^\delta(x) := \frac{(1-\eta)\,\|u^\delta - F(x)\|^2 - \delta\,\|u^\delta - F(x)\|\,(1+\eta)}{\|s_k^\delta(x)\|^2}, \tag{16}$$

where η is a nonlinearity parameter. As a stopping rule, we employ the well-known **discrepancy principle**. In order to speed up the iteration, we employ **Nesterov's acceleration strategy**, i.e., we use the modified iteration

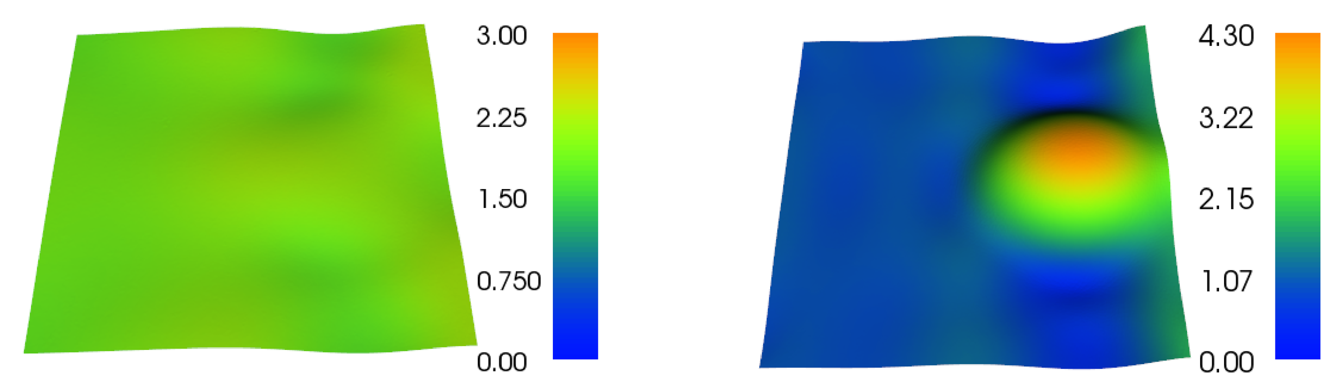
$$z_k^\delta = x_k^\delta + \tfrac{k-1}{k+2}\left(x_k^\delta - x_{k-1}^\delta\right), \qquad x_{k+1}^\delta = x_k^\delta + \omega_k^\delta(z_k^\delta)\, s_k^\delta(z_k^\delta). \tag{17}$$

Numerical Results

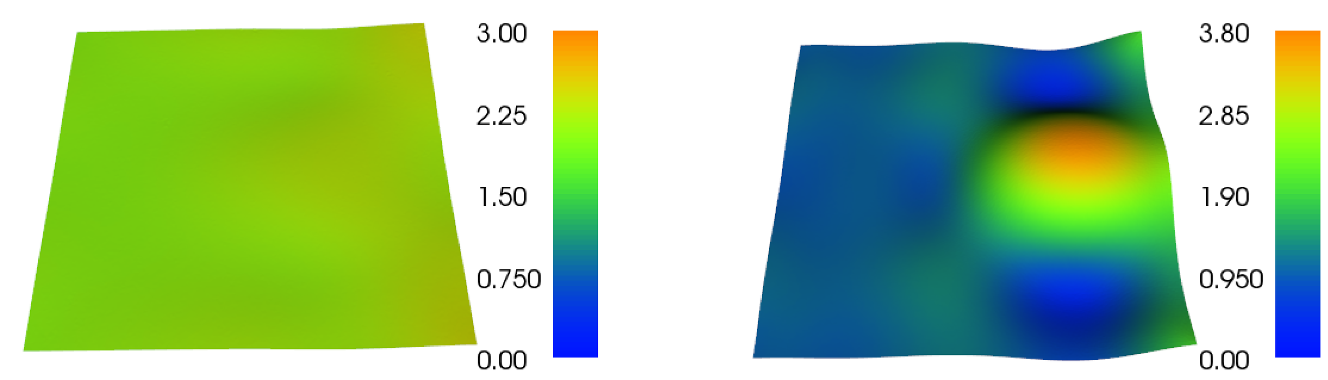
Motivated by the physical application, we consider a square object consisting of two linearly elastic isotropic materials. On the top edge, the object is fixed, i.e., $g_D = 0$, and on the left and right edges, the boundary remains free to move, which corresponds to $g_T = 0$. A constant unidirectional upward displacement $g_D = (0, 0.1)$ is being applied at the bottom edge and we assume that no body forces are present, i.e., $f = 0$. The spatial distributions of the exact Lamé parameters λ (uniform background of value 2) and μ (background of value 1 with inclusion of value 5) are depicted in the following figures:



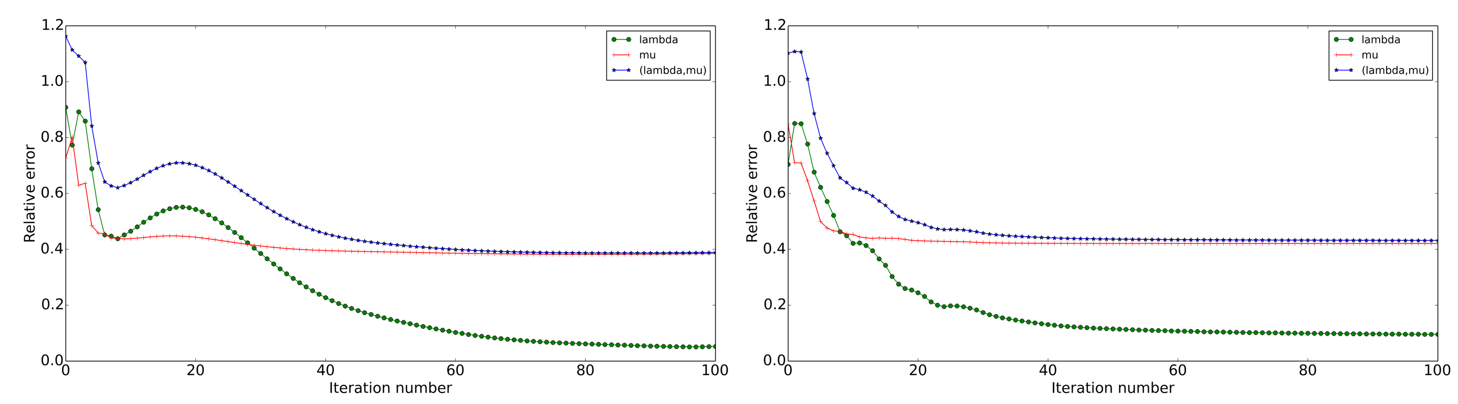
Reconstruction with steepest descent stepsize (15) and Nesterov acceleration (1% noise):



Reconstruction with Neubauer's new stepsize (16) and Nesterov acceleration (1% noise):



Relative error measured in the $L^2(\Omega)$ norm:



Steepest descent stepsize (15) Neubauer's new stepsize (16)

Conclusions & Outlook

- We proposed an operator formulation for the nonlinear inverse problem of linearized elasticity and presented numerical simulations based on Landweber type gradient methods combined with Nesterov acceleration.
- A concise convergence analysis of the employed algorithms as well as their improvement and application to real world problems will be topics of future research.

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