ON THE INVERSE PROBLEM OF LINEARIZED ELASTICITY

Ekaterina Sherina[1], Simon Hubmer[2]

[1]Technical University of Denmark, Department of Applied Mathematics and Computer Science, Kongens Lyngby, Denmark (sershe@dtu.dk)
[2]Johannes Kepler University Linz, Doctoral Program Computational Mathematics, Linz, Austria (simon.hubmer@dk-compmath.jku.at)

Introduction and Motivation

Elastography is an imaging modality that can distinguish materials by their biomechanical properties [1]. Doctors use palpation to detect abnormal tissues, which motivates the development of quantitative elasticity imaging.

Problem: Identify the Lamé parameters from displacement measurements.

Mathematical Model

Given a bounded, open and connected set $\Omega \in \mathbb{R}^N$, $N = 2,3$, with Lipschitz continuous boundary $\partial\Omega = \overline{\Gamma_D \cup \Gamma_T}$, $\Gamma_D \cap \Gamma_T = \emptyset$, $\operatorname{meas}(\Gamma_D) > 0$ and body forces f, prescribed displacement g_D, surface traction g_T and **Lamé parameters** λ and μ, the homogenized equations of **linearized elasticity** with displacement-traction boundary conditions are given by

$$\begin{aligned} -\operatorname{div}(\sigma(u)) &= f + \operatorname{div}(\sigma(\Phi)), \quad \text{in } \Omega, \\ u|_{\Gamma_D} &= 0, \\ \sigma(u)\vec{n}|_{\Gamma_T} &= g_T - \sigma(\Phi)\vec{n}|_{\Gamma_T}, \end{aligned} \tag{1}$$

where $\vec{n}$ is an outward unit normal, Φ is a function such that $\Phi|_{\Gamma_D} = g_D$, **the strain tensor** $\mathcal{E}$ and **the stress tensor** σ defining the stress-strain relation in Ω are given by

$$\mathcal{E}(u) := \frac{1}{2}\left(\nabla u + \nabla u^T\right), \qquad \sigma(u) := 2\mu\,\mathcal{E}(u) + \lambda \operatorname{div}(u)\, I. \tag{2}$$

Inverse Problem

The linearized elasticity problem (1) in the weak form:

$$a_{\lambda,\mu}(u,v) = l(v) - a_{\lambda,\mu}(\Phi, v), \qquad \forall\, v \in V, \tag{3}$$

with the linear and the bilinear forms

$$l(v) := \langle f, v \rangle + \langle g_T, v \rangle, \tag{4}$$

$$a_{\lambda,\mu}(u,v) := \int_\Omega \left(\lambda \operatorname{div}(u)\operatorname{div}(v) + 2\mu\,\mathcal{E}(u) : \mathcal{E}(v)\right) dx. \tag{5}$$

The nonlinear operator called **parameter-to-solution map** is

$$\begin{aligned} F : \mathcal{D}(F) := \left\{(\lambda,\mu) \in H^2(\Omega)^2 \,|\, \lambda \geq 0,\, \mu \geq \underline{\mu} > 0\right\} \subset H^2(\Omega)^2 &\to L^2(\Omega)^N, \\ (\lambda,\mu) &\mapsto u(\lambda,\mu), \end{aligned} \tag{6}$$

where $u(\lambda,\mu)$ is the solution of (3), then our inverse problem is:

Problem. Given f, g_D, Φ, g_T and a measurement u^δ of the true displacement field u satisfying $\left\| u - u^\delta \right\| \leq \delta$, compute an approximation of the **Lamé parameters** λ and μ, which satisfy

$$F(\lambda,\mu) = u. \tag{7}$$

Derivative and Adjoint

In the alternative form:

$$F(\lambda,\mu) = A^{-1}_{\lambda,\mu}\left(l - \tilde{A}_{\lambda,\mu}\Phi\right), \tag{8}$$

where the operator $\tilde{A}_{\lambda,\mu}$ connected to the bilinear form $a_{\lambda,\mu}$ by

$$\tilde{A}_{\lambda,\mu} : u \mapsto \left(v \mapsto a_{\lambda,\mu}(u,v)\right), \tag{9}$$

and A is its restriction to V, i.e., $A := \tilde{A}|_V$.

Theorem 1. *F is a well-defined, continuously Fréchet differentiable operator satisfying*

$$F'(\lambda,\mu)(h_\lambda, h_\mu) = -A^{-1}_{\lambda,\mu}\left(A_{h_\lambda,h_\mu} u(\lambda,\mu) + \tilde{A}_{h_\lambda,h_\mu}\Phi\right). \tag{10}$$

Theorem 2. *The adjoint of the Fréchet derivative of F is given by*

$$F'(\lambda,\mu)^* w = \begin{pmatrix} E\left(\operatorname{div}\left(u(\lambda,\mu) + \Phi\right)\operatorname{div}\left(-A^{-1}_{\lambda,\mu} T w\right)\right) \\ E\left(2\mathcal{E}\left(u(\lambda,\mu) + \Phi\right) : \mathcal{E}\left(-A^{-1}_{\lambda,\mu} T w\right)\right) \end{pmatrix}^T, \tag{11}$$

where T and E are defined by

$$T : w \mapsto \left(v \mapsto \int_\Omega w \cdot v \, dx\right), \qquad \langle Eu, v \rangle = \int_\Omega u v \, dx.$$

Reconstruction Algorithm

Landweber type gradient methods with the abbreviation $x_k^\delta = \left(\lambda_k^\delta, \mu_k^\delta\right)$:

$$x_{k+1}^\delta = x_k^\delta + \omega_k^\delta\left(x_k^\delta\right) s_k^\delta\left(x_k^\delta\right), \qquad s_k^\delta(x) := F'(x)^*\left(u^\delta - F(x)\right), \tag{12}$$

where for the stepsize ω_k^δ we use both the **steepest descent** stepsize

$$\omega_k^\delta(x) := \left\| s_k^\delta(x) \right\|^2 / \left\| F'(x) s_k^\delta(x) \right\|^2, \tag{13}$$

and the recently introduced [2] stepsize

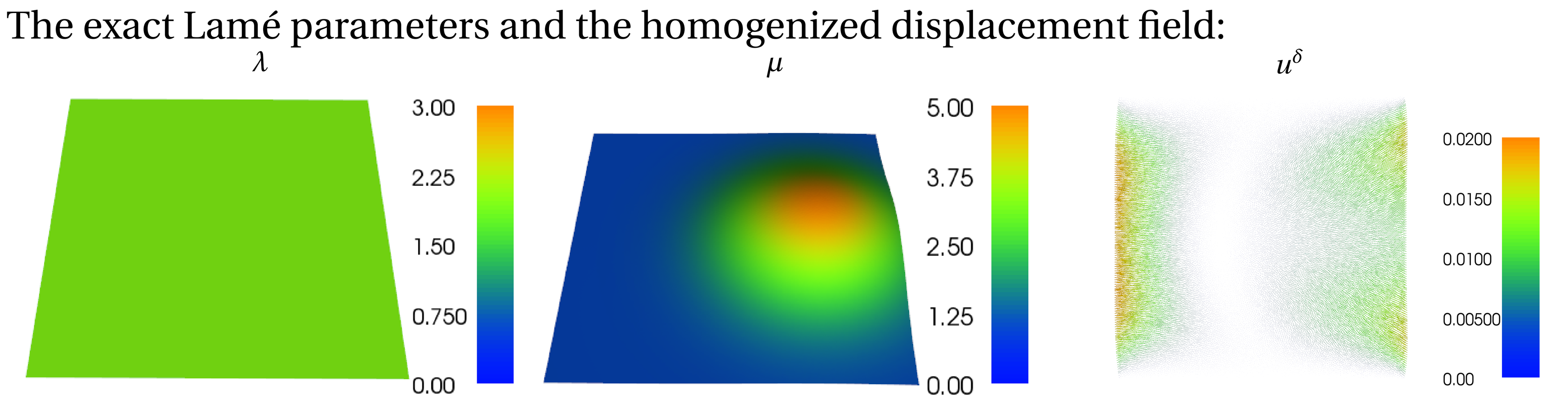
$$\omega_k^\delta(x) := \left((1-\eta)\left\| u^\delta - F(x) \right\|^2 - \delta \left\| u^\delta - F(x) \right\| (1+\eta)\right) / \left\| s_k^\delta(x) \right\|^2, \tag{14}$$

where η is a nonlinearity parameter. As a stopping rule, we employ the well-known **discrepancy principle**. For speeding up the methods, we employ **Nesterov's acceleration strategy**, the modified iteration is:

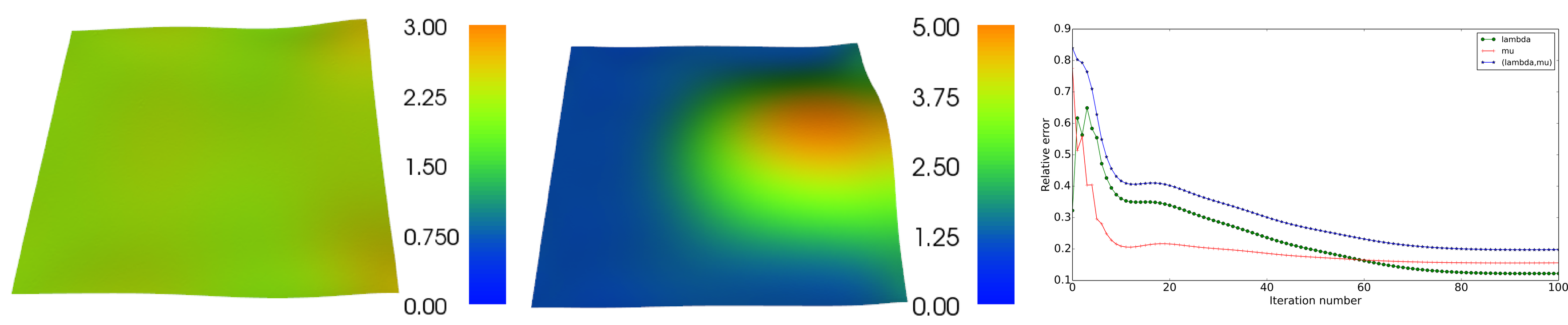
$$z_k^\delta = x_k^\delta + \tfrac{k-1}{k+2}\left(x_k^\delta - x_{k-1}^\delta\right), \qquad x_{k+1}^\delta = x_k^\delta + \omega_k^\delta\left(z_k^\delta\right) s_k^\delta\left(z_k^\delta\right). \tag{15}$$

Numerical Results

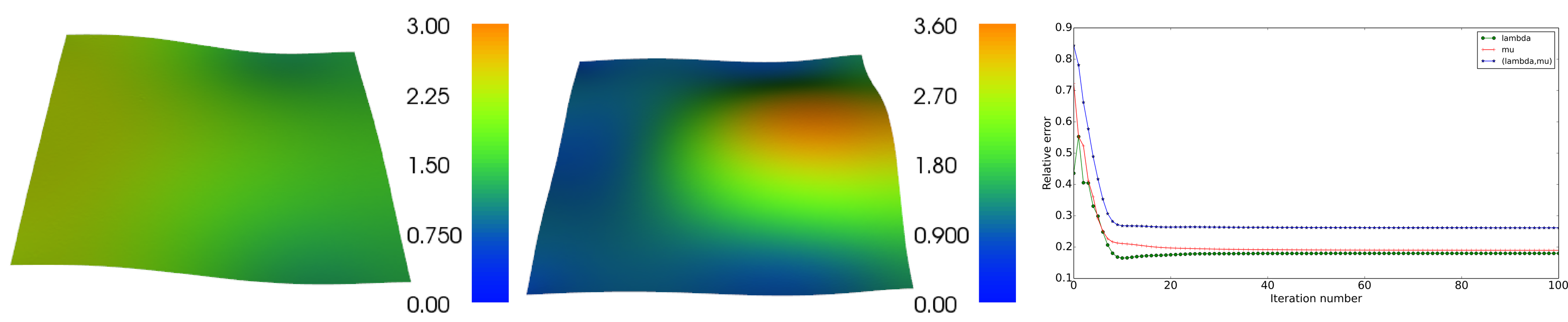
The exact Lamé parameters and the homogenized displacement field:



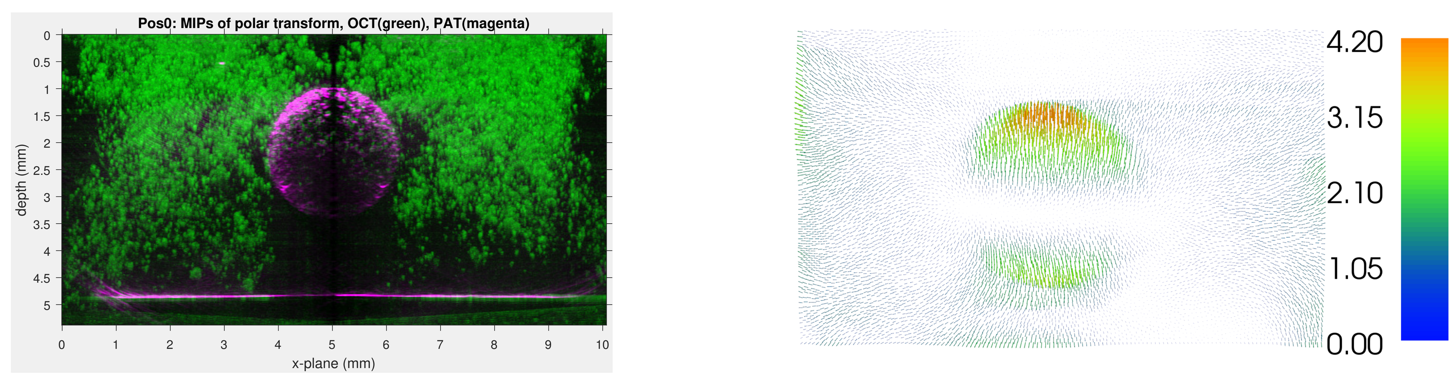
Reconstruction with steepest descent stepsize (13) and Nesterov acceleration (1% noise):



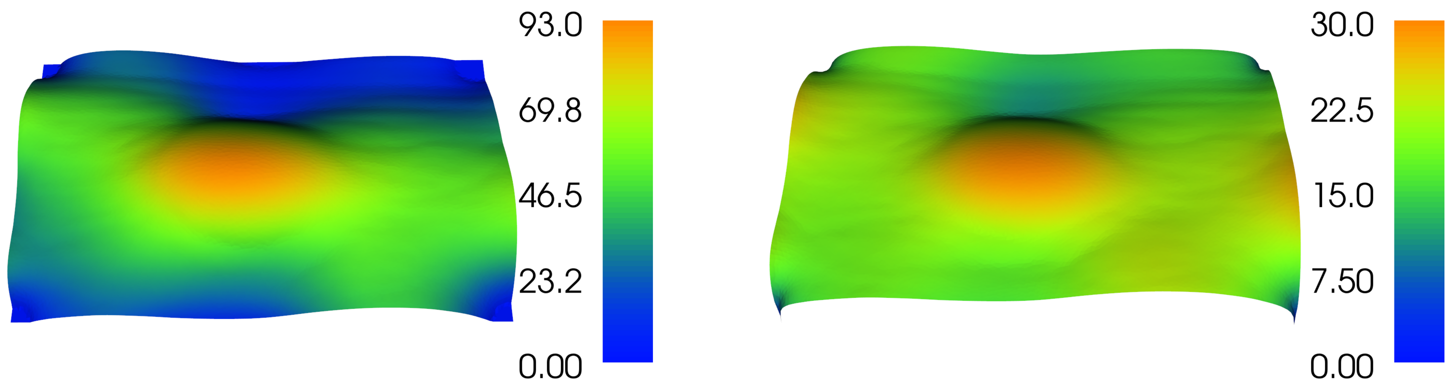
Reconstruction with Neubauer's new stepsize (14) and Nesterov acceleration (1% noise):



The displacement field from the physical experiment:



Reconstruction with steepest descent stepsize (13) and Nesterov acceleration:



Conclusions & Outlook

- We proposed an operator formulation for the nonlinear inverse problem of linearized elasticity and presented numerical simulations based on Landweber type gradient methods combined with Nesterov acceleration.
- A concise convergence analysis of the employed algorithms as well as their improvement and application to further real world problems will be topics of future research.

Acknowledgement

The authors were funded by the Austrian Science Fund (FWF): W1214-N15, project DK8, and by the Danish Council for Independent Research - Natural Sciences: grant 4002-00123. Furthermore, they would like to thank Prof. Otmar Scherzer, Prof. Andreas Neubauer, Dr. Stefan Kindermann, Prof. Walter Zulehner and Prof. Ulrich Langer.

References

[1] M. M. Doyley. Model-based elastography: a survey of approaches to the inverse elasticity problem. *Physics in Medicine and Biology*, 57(3), 2012.
[2] A. Neubauer. A new gradient method for ill-posed problems. 2017. submitted.