Inverse Problems and MRAI

Mapping the pulse wave velocity

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

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Introduction

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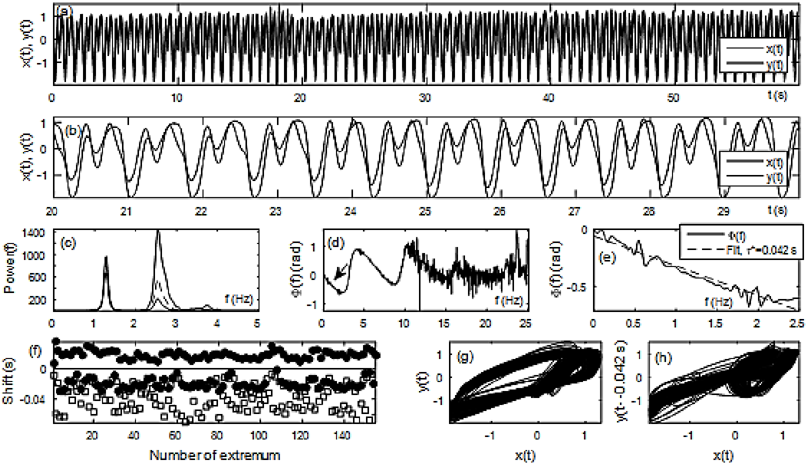
- What is the PWV?
- Why do we want to estimate it?
- How can we estimate it?

Pulse Wave Velocity



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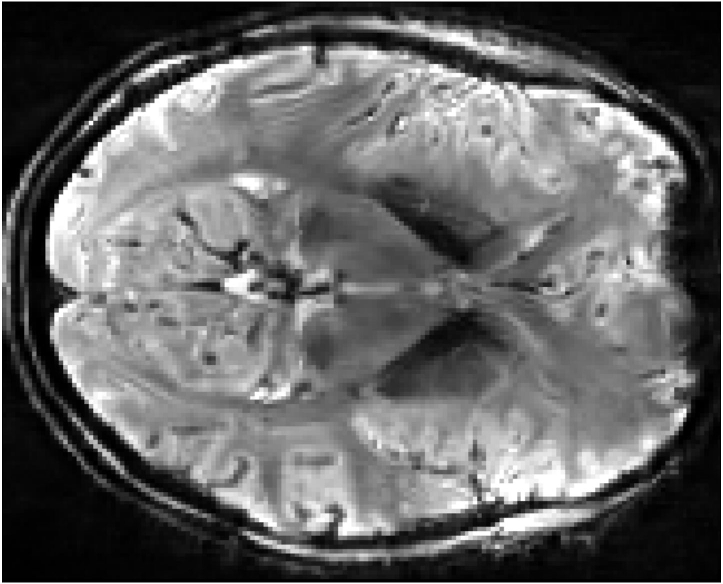
The parameters are

- h - vessel wall thickness,
- d - vascular diameter,
- E - vessel all's Young modulus,
- ρ_B - density of the blood.

Introduction and Motivation



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- pulse wave velocity $v(x, y, z)$.

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MRAI = Magnetic Resonance Advection Imaging

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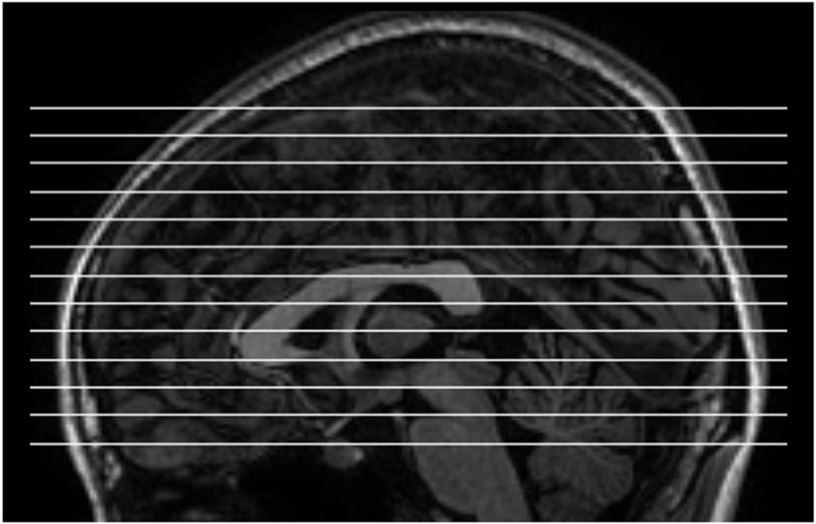
- Partial data $\longleftrightarrow$ slice-time-acquisition problem.

Slice-Time Acquisition



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Idea: Discretize the advection equation according to the data!!

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The continuous advection equation

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then becomes a discrete linear system of equations:

$$\frac{\rho_{i,j,k,l}-\rho_{i,j,k,l-1}}{(K+1)\Delta t} + D_{x_i}\rho_{i,j,k,l}\, v_{1,i,j,k} + D_{y_j}\rho_{i,j,k,l}\, v_{2,i,j,k} + D_{z_k}\rho_{i,j,k,l}\, v_{3,i,j,k} = 0\,,$$

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where

$$\rho_{i,j,k,l} = \rho(x_i,y_j,z_k,t_{k,l})\,,\quad v_{m,i,j,k} = v_m(x_i,y_j,z_k)\,,\quad m=1,2,3\,.$$

Discretization - Finite Differences

$$D_{x_i}\rho_{i,j,k,l} := \begin{cases} \dfrac{\rho_{i+1,j,k,l} - \rho_{i-1,j,k,l}}{2\Delta x}, & 1 \leq i \leq I-1 \\ \dfrac{\rho_{1,j,k,l} - \rho_{0,j,k,l}}{\Delta x}, & i = 0 \\ \dfrac{\rho_{I,j,k,l} - \rho_{I-1,j,k,l}}{\Delta x}, & i = I \end{cases}$$

$$D_{y_j}\rho_{i,j,k,l} := \begin{cases} \dfrac{\rho_{i,j+1,k,l} - \rho_{i,j-1,k,l}}{2\Delta y}, & 1 \leq j \leq J-1 \\ \dfrac{\rho_{i,1,k,l} - \rho_{i,0,k,l}}{\Delta y}, & j = 0 \\ \dfrac{\rho_{i,J,k,l} - \rho_{i,J-1,k,l}}{\Delta y}, & j = J \end{cases}$$

Discretization - Finite Differences

$$D_{z_k}\rho_{i,j,k,l} := \begin{cases} \dfrac{(1-r)(\rho_{i,j,k+1,l}-\rho_{i,j,k-1,l+1}) + r(\rho_{i,j,k+1,l-1}-\rho_{i,j,k-1,l})}{2\Delta z}, & 1 \leq k \leq K-1, 1 \leq l < L \\ \dfrac{(1-r)\rho_{i,j,k+1,L} - (1+r)\rho_{i,j,k-1,L} + r(\rho_{i,j,k+1,L-1}+\rho_{i,j,k-1,L-1})}{2\Delta z}, & 1 \leq k \leq K-1, l = L \\ \dfrac{(1-r)\rho_{i,j,1,l} + r\rho_{i,j,1,l-1} - \rho_{i,j,0,l}}{\Delta z}, & k = 0, 1 \leq l \leq L \\ \dfrac{\rho_{i,j,K,l} - (1-r)\rho_{i,j,K-1,l+1} - r\rho_{i,j,K-1,l}}{\Delta z}, & k = K, 1 \leq l < L \\ \dfrac{\rho_{i,j,K,L} - (1+r)\rho_{i,j,K-1,L} + r\rho_{i,j,K-1,L-1}}{\Delta z}, & k = K, l = L \end{cases}$$

$$r := \frac{1}{K+1}$$

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We denote the solution $\vec{\rho}$ of this equation with $\rho(\vec{v}, \vec{\rho}_0)$.

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where the inner products on $\mathcal{X}$ and $\mathcal{Y}$ are given by

$$\langle \, (\vec{v}, \vec{\rho}_0), (\vec{x}, \vec{w}_0) \, \rangle_{\mathcal{X}} := \vec{v}^{\,T} H \vec{x} + \vec{\rho}_0^{\,T} \vec{w}_0 \,,$$
$$\langle \, (\vec{\rho}, \vec{\rho}_0), (\vec{w}, \vec{w}_0) \, \rangle_{\mathcal{Y}} := \vec{\rho}^{\,T} \vec{w} + \vec{\rho}_0^{\,T} \vec{w}_0 \,.$$

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We can now write our problem in standard form, i.e.,

$$\boxed{\text{"}\; F\left(\vec{v}, \vec{\rho}_0\right) = \left(\vec{\rho}^{\,\delta}, \vec{\rho}_0^{\,\delta}\right) \;\text{"}}$$

Derivative and Adjoint

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The Frechet derivative is given by

$$F'(\vec{v}, \vec{\rho}_0)(\Delta\vec{v}, \Delta\vec{\rho}_0) = (\rho'(\vec{v}, \vec{\rho}_0)(\Delta\vec{v}, \Delta\vec{\rho}_0), \Delta\vec{\rho}_0)\,,$$

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It's adjoint is given by

$$F'(\vec{v}, \vec{\rho}_0)^*(\vec{w}, \vec{w}_0) = \begin{pmatrix} H^{-1}\left(-D_A(\vec{v}, \rho(\vec{v}, \vec{\rho}_0))^T + b'_{\Delta\vec{\rho}_0}(\vec{v}, \vec{\rho}_0)^T\right) A(\vec{v})^{-T}\vec{w} \\ b'_{\Delta\vec{v}}(\vec{v}, \vec{\rho}_0)^T A(\vec{v})^{-T}\vec{w} + \vec{w}_0 \end{pmatrix}.$$

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$$\Longrightarrow F(\vec{v},\vec{\rho}_0) := (\rho(\vec{v},\vec{\rho}_0), \vec{\rho}_0, D\vec{v})\,.$$

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Remember the inner product:

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- **Solution 2:** Use Wavelets instead of H.

Computation Method

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Landweber type gradient method:

$$x_{k+1}^{\delta} = x_k^{\delta} + \omega_k^{\delta} \, F'(x_k^{\delta})^* (y^{\delta} - F(x_k^{\delta}))\,.$$

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Steepest descent stepsize:

$$\omega_k^{\delta} = \frac{\left\| s_k \right\|^2}{\left\| F'(x_k^{\delta}) s_k^{\delta} \right\|^2}\,, \qquad s_k^{\delta} = F'(x_k^{\delta})^*(y^{\delta} - F(x_k^{\delta}))\,.$$

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Discrepancy principle:

$$\left\| y^\delta - F(x_{k_*}^\delta) \right\| \leq \tau\delta \leq \left\| y^\delta - F(x_k^\delta) \right\| \, , \quad 0 \leq k \leq k_* \, .$$

Sparsity

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Shrinkage function:

$$S_{\tau,p}(x) = \begin{cases} \operatorname{sgn}(x)\max(|x| - \tau, 0)\,, & p = 1\,, \\ G_{\tau,p}^{-1}(x)\,, & p \in (1,2]\,, \end{cases}$$

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Resulting iteration:

$$x_{k+1}^\delta = S_{\omega_k^\delta \alpha,p}\left(x_k^\delta + \omega_k^\delta\, F'(x_k^\delta)^*(y^\delta - F(x_k^\delta))\right)\ .$$

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- **Essential:** Stefan Engblom's *fsparse.m* file.

Implementation Details

- **Software:** MATLAB R2015b.
- **Solver:** biCGstab with iLU preconditioner.
- **Parallelization:** As far as possible.
- **Essential:** Stefan Engblom's *fsparse.m* file.

$\Longrightarrow$ *Runs on a standard home computer in acceptable time!!!*

(Real-world data set has 3 million unknowns)

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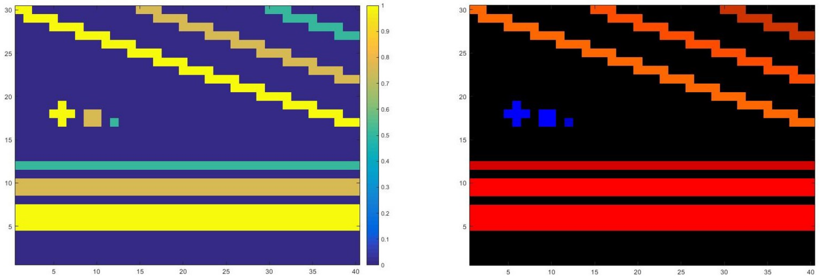
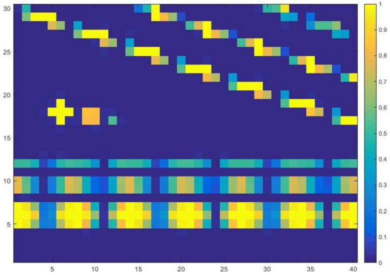
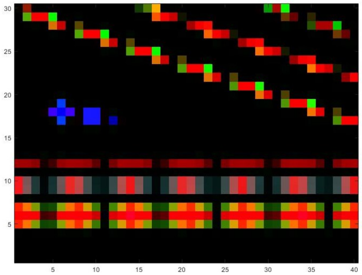
$\Longrightarrow$ Run the algorithm using the discrepancy principle ($\tau = 1.1$).

Simulation Phantom - MIP and direction MIP

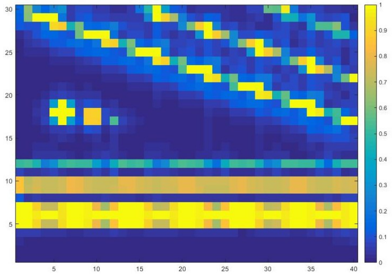


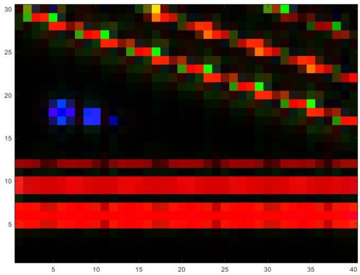
Results - Pure Method



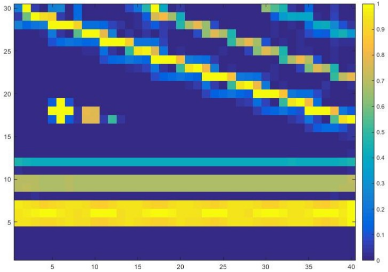


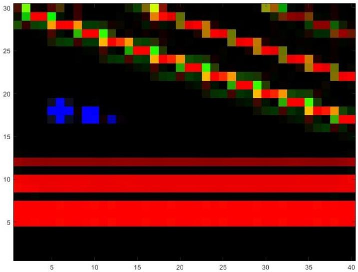
Results - Divergence-Free





Results - Divergence-Free + Sparsity





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Algorithm specifics:

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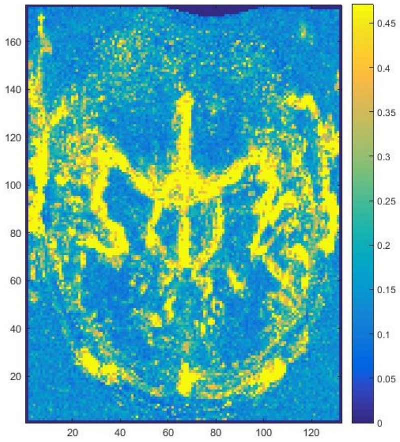
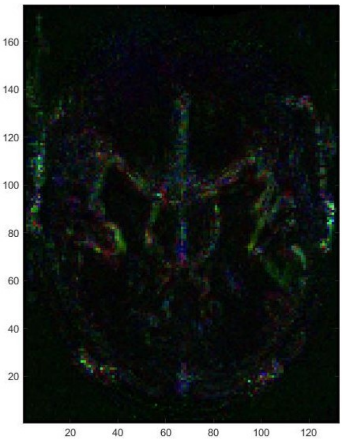
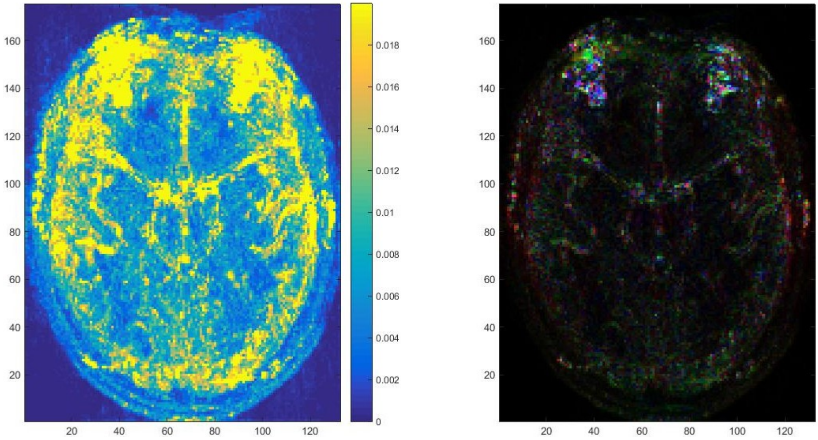
- **Stopping rule:** Residual decrease check.

Regression Approach - Results





New Approach - Results



End

Thank you for your attention!