Pulse Wave Velocity Estimation

via Magnetic Resonance Advection Imaging

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

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Introduction

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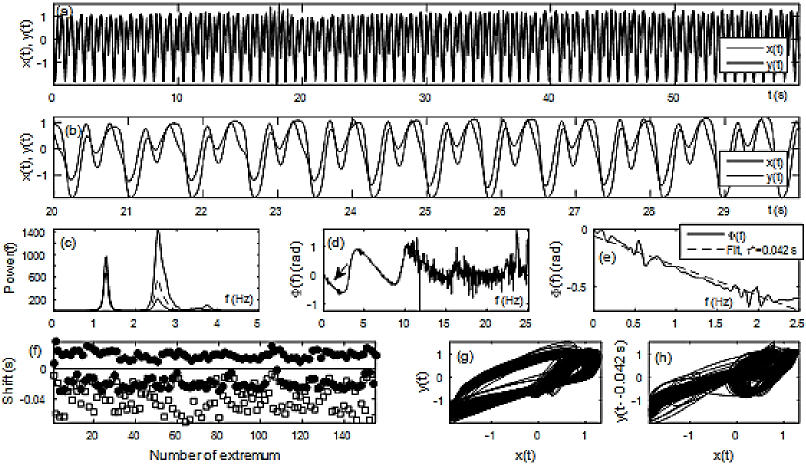
Three natural questions:

- What is the PWV?
- Why do we want to estimate it?
- How can we estimate it?

The Pulse Wave



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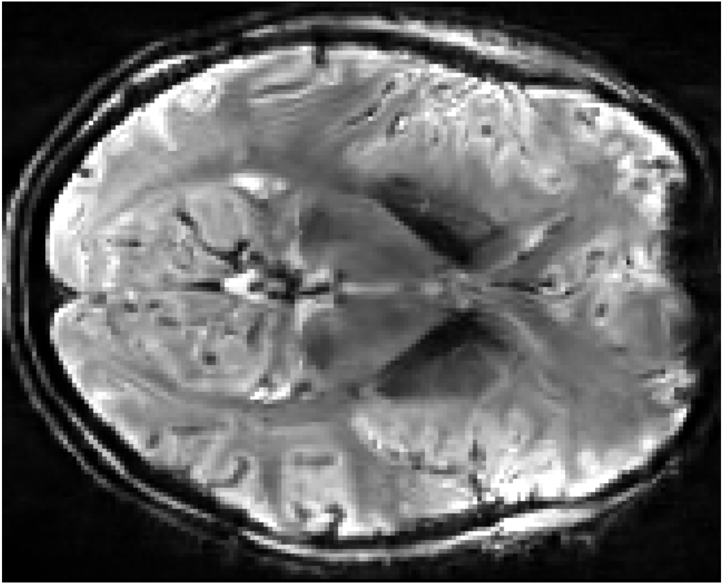
The parameters are

- h - vessel wall thickness,
- d - vascular diameter,
- E - vessel all's Young modulus,
- ρ_B - density of the blood.

Introduction and Motivation



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Problem variables

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$$\boxed{\frac{\partial}{\partial t}\rho(x, y, z, t) + \operatorname{div}\left(v(x, y, z)\rho(x, y, z, t)\right) = 0\,.}$$

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MRAI = Magnetic Resonance Advection Imaging

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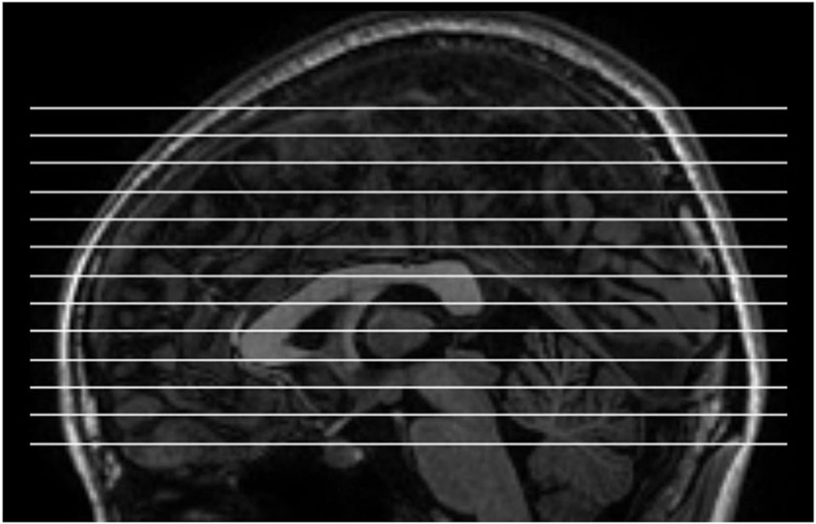
- Partial data $\longleftrightarrow$ slice-time-acquisition problem.

Slice-Time Acquisition



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$$x_i = x_0 + i\Delta x\,, \quad y_j = y_0 + j\Delta y\,, \quad z_k = z_0 + k\Delta z\,,$$
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Idea: Discretize the advection equation according to the data!!

Solution Strategy - Discretization

The continuous advection equation

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then becomes a discrete linear system of equations:

$$\frac{\rho_{i,j,k,l}-\rho_{i,j,k,l-1}}{(K+1)\Delta t} + D_{x_i}\rho_{i,j,k,l}\,v_{1,i,j,k} + D_{y_j}\rho_{i,j,k,l}\,v_{2,i,j,k} + D_{z_k}\rho_{i,j,k,l}\,v_{3,i,j,k} = 0\,,$$

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where

$$\rho_{i,j,k,l} = \rho(x_i, y_j, z_k, t_{k,l})\,, \quad v_{m,i,j,k} = v_m(x_i, y_j, z_k)\,, \quad m = 1,2,3\,.$$

Discretization - Finite Differences

$$D_{x_i}\rho_{i,j,k,l} := \begin{cases} \dfrac{\rho_{i+1,j,k,l} - \rho_{i-1,j,k,l}}{2\Delta x}, & 1 \leq i \leq I-1 \\ \dfrac{\rho_{1,j,k,l} - \rho_{0,j,k,l}}{\Delta x}, & i = 0 \\ \dfrac{\rho_{I,j,k,l} - \rho_{I-1,j,k,l}}{\Delta x}, & i = I \end{cases}$$

$$D_{y_j}\rho_{i,j,k,l} := \begin{cases} \dfrac{\rho_{i,j+1,k,l} - \rho_{i,j-1,k,l}}{2\Delta y}, & 1 \leq j \leq J-1 \\ \dfrac{\rho_{i,1,k,l} - \rho_{i,0,k,l}}{\Delta y}, & j = 0 \\ \dfrac{\rho_{i,J,k,l} - \rho_{i,J-1,k,l}}{\Delta y}, & j = J \end{cases}$$

Discretization - Finite Differences

$$D_{z_k}\rho_{i,j,k,l} := \begin{cases} \dfrac{(1-r)(\rho_{i,j,k+1,l}-\rho_{i,j,k-1,l+1}) + r(\rho_{i,j,k+1,l-1}-\rho_{i,j,k-1,l})}{2\Delta z}, & 1 \leq k \leq K-1, 1 \leq l < L \\ \dfrac{(1-r)\rho_{i,j,k+1,L} - (1+r)\rho_{i,j,k-1,L} + r(\rho_{i,j,k+1,L-1}+\rho_{i,j,k-1,L-1})}{2\Delta z}, & 1 \leq k \leq K-1, l = L \\ \dfrac{(1-r)\rho_{i,j,1,l} + r\rho_{i,j,1,l-1} - \rho_{i,j,0,l}}{\Delta z}, & k = 0, 1 \leq l \leq L \\ \dfrac{\rho_{i,j,K,l} - (1-r)\rho_{i,j,K-1,l+1} - r\rho_{i,j,K-1,l}}{\Delta z}, & k = K, 1 \leq l < L \\ \dfrac{\rho_{i,j,K,L} - (1+r)\rho_{i,j,K-1,L} + r\rho_{i,j,K-1,L-1}}{\Delta z}, & k = K, l = L \end{cases}$$

$$r := \frac{1}{K+1}$$

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We denote the solution $\vec{\rho}$ of this equation with $\rho(\vec{v}, \vec{\rho}_0)$.

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$$\langle\, (\vec{v}, \vec{\rho}_0), (\vec{x}, \vec{w}_0)\,\rangle_{\mathcal{X}} := \vec{v}^{\,T} H \vec{x} + \vec{\rho}_0^{\,T} \vec{w}_0\,,$$
$$\langle\, (\vec{\rho}, \vec{\rho}_0), (\vec{w}, \vec{w}_0)\,\rangle_{\mathcal{Y}} := \vec{\rho}^{\,T} \vec{w} + \vec{\rho}_0^{\,T} \vec{w}_0\,.$$

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We can now write our problem in standard form, i.e.,

$$\text{"}\; F\,(\vec{v}, \vec{\rho}_0) = \left(\vec{\rho}^{\,\delta}, \vec{\rho}_0^{\,\delta}\right) \;\text{"}$$

Derivative and Adjoint

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The Frechet derivative is given by

$$F'(\vec{v}, \vec{\rho}_0)(\Delta\vec{v}, \Delta\vec{\rho}_0) = (\rho'(\vec{v}, \vec{\rho}_0)(\Delta\vec{v}, \Delta\vec{\rho}_0), \Delta\vec{\rho}_0)\,,$$

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It's adjoint is given by

$$F'(\vec{v},\vec{\rho}_0)^*(\vec{w},\vec{w}_0) = \begin{pmatrix} H^{-1}\left(-D_A(\vec{v},\rho(\vec{v},\vec{\rho}_0))^T + b'_{\Delta\vec{\rho}_0}(\vec{v},\vec{\rho}_0)^T\right)A(\vec{v})^{-T}\vec{w} \\ b'_{\Delta\vec{v}}(\vec{v},\vec{\rho}_0)^T A(\vec{v})^{-T}\vec{w} + \vec{w}_0 \end{pmatrix}.$$

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$$\Longrightarrow F(\vec{v},\vec{\rho}_0) := (\rho(\vec{v},\vec{\rho}_0), \vec{\rho}_0, D\vec{v})\,.$$

Choosing the matrix H

Remember the inner product:

$$\langle (\vec{v}, \vec{\rho}_0), (\vec{x}, \vec{w}_0) \rangle_{\mathcal{X}} = \vec{v}^{\,T} H \vec{x} + \vec{\rho}_0^{\,T} \vec{w}_0 \,.$$

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- **Solution 1:** Use only the diagonal entries.
- **Solution 2:** Use Wavelets instead of H.

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- Existence and uniqueness of solutions.
- How to approximate/compute particular solutions.
- *How to do it in an efficient way!!*

Nonlinear Inverse Problems and TPG methods

- Tikhonov Regularization (suitable α)

$$\min_x \left\{ \frac{1}{2} \left\| F(x) - y^\delta \right\|^2 + \frac{\alpha}{2} \left\| x - x_0 \right\|^2 \right\} .$$

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- Landweber Iteration (suitable stopping rule)

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- Iteratively regularized Gauss-Newton method

$$x_{k+1}^\delta = x_k^\delta + (F'(x_k^\delta)^* F'(x_k^\delta) + \alpha I)^{-1} (F'(x_k^\delta)^*(y^\delta - F(x_k^\delta)) + \alpha_k (x_0 - x_k^\delta)) .$$

Connection: Residual Functional

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- Tikhonov = Minimize{ $\Phi(x)$ + Regularization(x) }.

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- Iteratively regularized Gauss-Newton
  = 2nd order descent for $\Phi(x)$ + Tikhonov Type Stabilization

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in every iteration step $\rightarrow$ difficult and takes time.

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Examples: Steepest Descent, Barzilai-Borwein, Neubauer.

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use the following iteration:

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This can be used to incorporate sparsity constraints via

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- Similar results also when using the discrepancy principle.

Andreas Neubauer, *On Nesterov Acceleration for Landweber Iteration of Linear Ill-Posed Problems*, in press

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Open: Convergence for nonlinear problems and $\lambda_k^\delta = \frac{k-1}{k+\alpha-1}$

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$$\left\| F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x}) \right\| \leq \eta \left\| F(x) - F(\tilde{x}) \right\| ,$$

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- via a backtracking algorithm.

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Reading suggestion: *Convergence Analysis of a Two-Point Gradient Method for Nonlinear Ill-Posed Problems*, Hubmer, Ramlau, submitted.

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Steepest descent stepsize:

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Discrepancy principle:

$$\left\|y^\delta - F(z_{k_*}^\delta)\right\| \leq \tau\delta < \left\|y^\delta - F(z_k^\delta)\right\|\,, \quad 0 \leq k \leq k_*\,.$$

Implementation Details

- **Software:** MATLAB R2015b.
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- **Essential:** Stefan Engblom's *fsparse.m* file.

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$\Longrightarrow$ *Runs on a standard home computer in acceptable time!!!*

(Real-world data set has 3 million unknowns)

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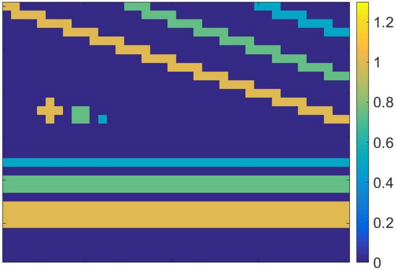
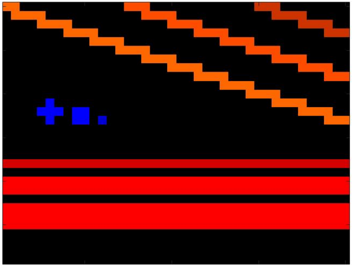
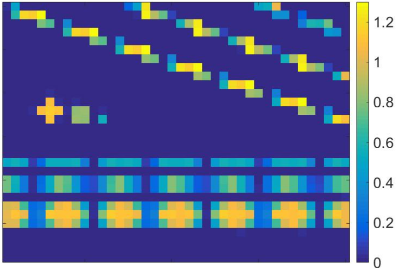
$\Longrightarrow$ Run the algorithm using the discrepancy principle ($\tau = 1.1$).

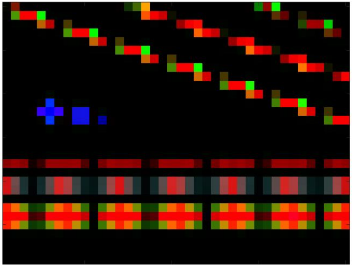
Simulation Phantom - MIP and direction MIP



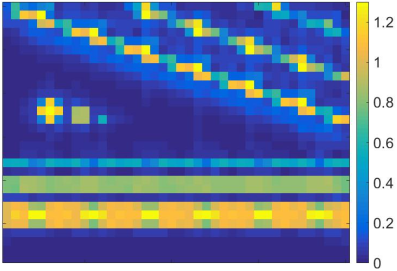


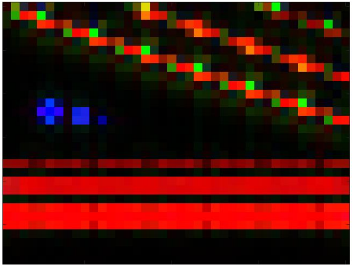
Results - Pure Method



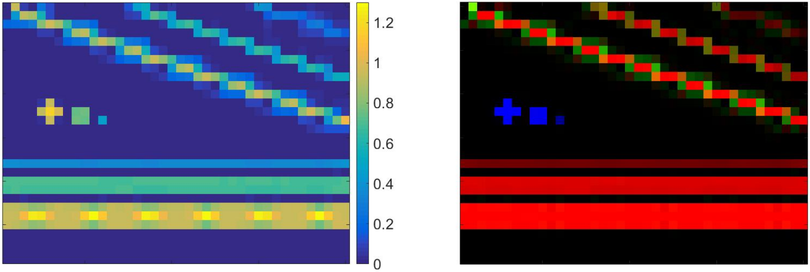


Results - Divergence-Free





Results - Divergence-Free + Wavelets + Sparsity



Natural Stimulation Data Set

Natural Stimulation Data Set

Specifications:

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- Publicly available natural stimulation dynamic EPI data.

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- Data has dimension $132 \times 175 \times 48$.

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- Pulse repetition time (TR) of 2 seconds.
- Eight 15 minutes long segments for each subject.

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Algorithm specifics:

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Algorithm specifics:

- First 20 seconds of second segment were used.

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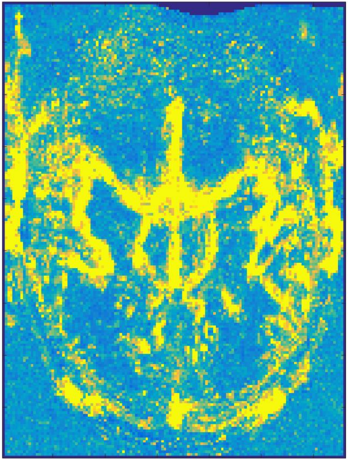
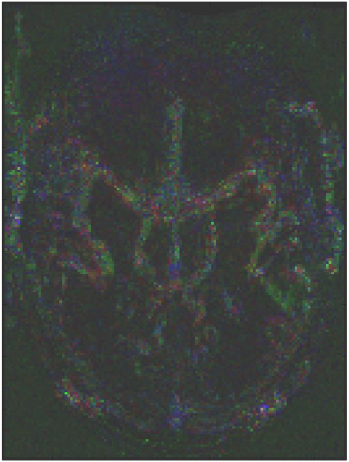
- Eight 15 minutes long segments for each subject.

Algorithm specifics:

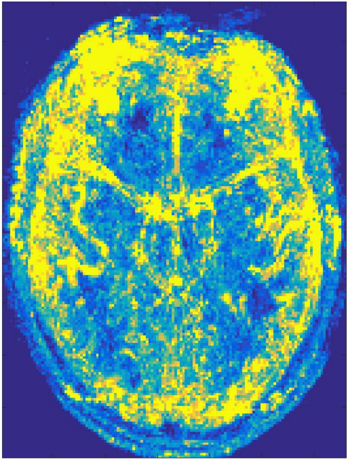
- First 20 seconds of second segment were used.
- **Stopping rule:** Residual decrease check.

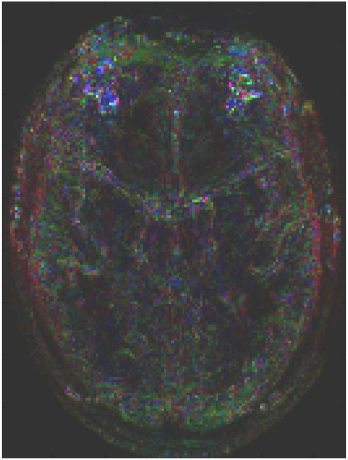
Regression Approach - Results



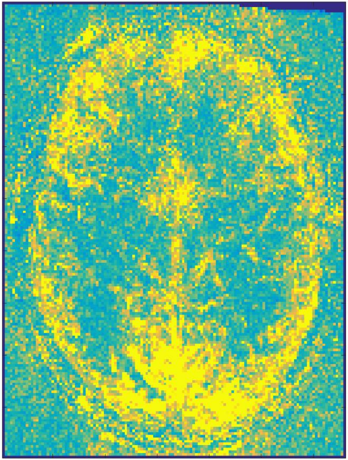


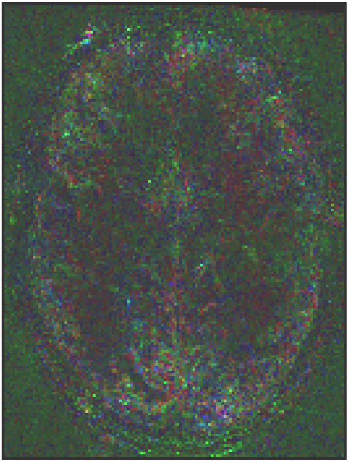
New Approach - Results



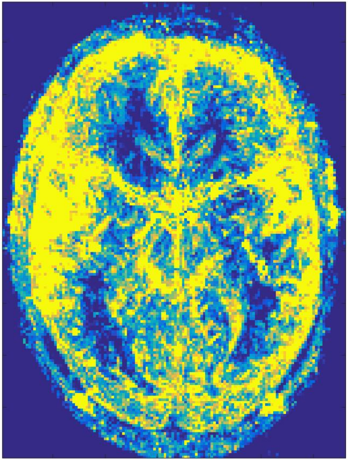


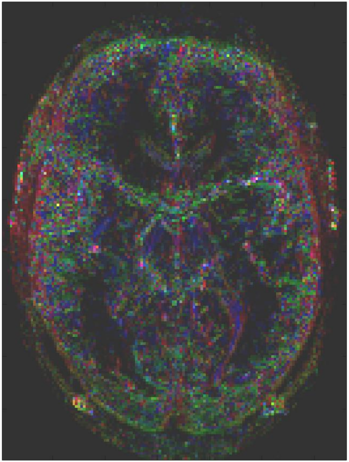
Regression Approach - Results





New Approach - Results





End

Thank you for your attention!