Two-Point Gradient Methods

for solving

Nonlinear Ill-Posed Problems

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Joint work with:

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

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The noisy data y^δ satisfies

$$\left\| y - y^\delta \right\| \leq \delta\,.$$

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Further examples: CT, MRI, MRAI, PI, EIT, AO, . . .

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− Computation of the minimum $\leftrightarrow$ HOW??

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Properties:

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– Strong conditions necessary for analysis.
– Slow convergence, i.e., lots of iterations required.

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$$x_{k+1}^{\delta} = x_k^{\delta} + (F'(x_k^{\delta})^* F'(x_k^{\delta}) + \alpha_k I)^{-1} F'(x_k^{\delta})^* (y^{\delta} - F(x_k^{\delta})) \,.$$

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Properties:

+ Require much less iterations.
– Very strong conditions necessary for analysis.
– Require inversion of $(F'(x)^* F'(x) + \alpha I)$ in every iteration step $\rightarrow$ difficult and takes time.

Acceleration Techniques

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- Landweber Iteration with operator approximation:

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- Landweber Iteration with intelligent stepsizes:

$$x_{k+1}^{\delta} = x_k^{\delta} + \alpha_k^{\delta} F'(x_k^{\delta})^*(y^{\delta} - F(x_k^{\delta}))\,.$$

Examples: Steepest Descent, Barzilai-Borwein, Neubauer.

Connection: Residual Functional

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- Levenberg Marquardt = 2nd order descent for $\Phi(x)$.
- Iteratively regularized Gauss-Newton
  = 2nd order descent for $\Phi(x)$ + Tikhonov Type Stabilization.

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General minimization problem

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Yurii Nesterov: Instead of using gradient descent:

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use the following iteration:

$$z_k = x_k + \tfrac{k-1}{k+\alpha-1}(x_k - x_{k-1}) ,$$
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Motivating Picture

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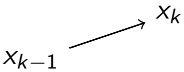

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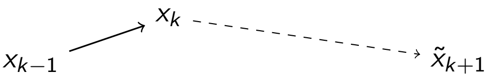

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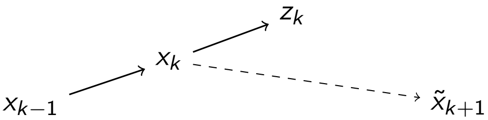

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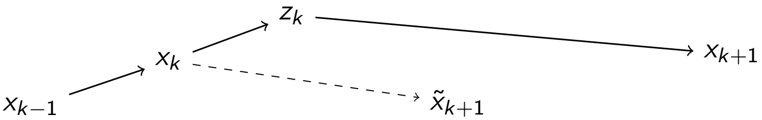

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H. Attouch, J. Peypouquet, The rate of convergence of Nesterov's accelerated forward-backward method is actually $o(k^{-2})$, SIAM Journal on Optimization

Y. Nesterov, A method of solving a convex programming problem with convergence rate $O(1/k^2)$, Soviet Mathematics Doklady, 27, 2, 1983

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$\Longrightarrow$ Sparsity Constraints, Projections, etc.

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A. Neubauer, On Nesterov acceleration for Landweber iteration of linear ill-posed problems, J. Inv. Ill-Posed Problems, to appear.

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Open: Convergence for nonlinear problems and $\lambda_k^\delta = \frac{k-1}{k+\alpha-1}$

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- Parameters $0 \leq \lambda_k^\delta \leq 1$ and stepsizes $\alpha_k^\delta \geq \alpha_{\min} > 0$

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- Parameters $0 \leq \lambda_k^\delta \leq 1$ and stepsizes $\alpha_k^\delta \geq \alpha_{\min} > 0$ satisfy

$$\lambda_k^\delta(\lambda_k^\delta + 1) \left\| x_k^\delta - x_{k+1}^\delta \right\|^2 - \left(1 + \frac{\Psi}{\mu}\right) \alpha_k^\delta \left\| F(z_k^\delta) - y^\delta \right\|^2$$

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Convergence Conditions

- Nonlinearity Condition

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- Parameters λ_k^δ satisfy

$$\sum_{k=0}^{\infty} \lambda_k^0 \left\| x_k^0 - x_{k-1}^0 \right\| < \infty \,.$$

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- via a backtracking algorithm.

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Discrepancy Principle:

$$\left\| y^\delta - F(z^\delta_{k_*}) \right\| \leq \tau\delta < \left\| y^\delta - F(z^\delta_k) \right\| , \qquad 0 \leq k < k_* = k_*(\delta, y^\delta) .$$

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Under the above assumptions, there holds

$$\lim_{\delta\to 0} z^\delta_{k_*(\delta,y^\delta)} = x_*\,.$$

If additionally $\mathcal{N}(F'(x^\dagger)) \subset \mathcal{N}(F'(x))$*, then we have*

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5. If $\mathcal{N}(F'(x^{\dagger})) \subset \mathcal{N}(F'(x))$, use a special property of $x^{\dagger}$.

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For many stepsizes α_k^δ, the *coupling condition* above reduces to

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Idea: Given a *summable* sequence $(q_n)_n$, choose λ_k^δ via

$$\lambda_k^\delta = \min \left\{ \frac{q_{n_k}}{\left\| x_k^\delta - x_{k-1}^\delta \right\|}, 1 \right\} ,$$

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$$\sum_{k=0}^{\infty} \lambda_k^0 \left\| x_k^0 - x_{k-1}^0 \right\| < \infty .$$

Example Problem: SPECT

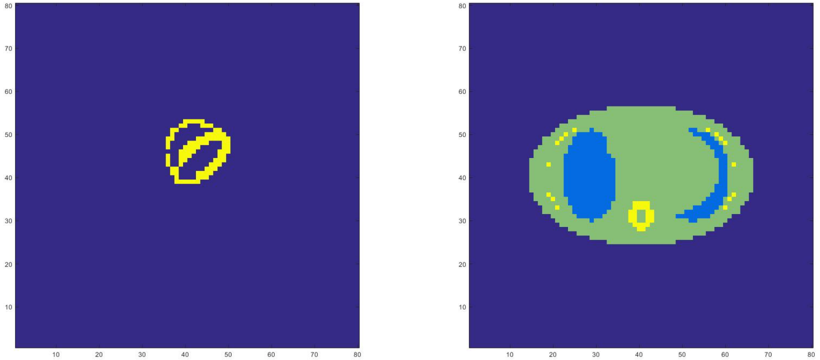
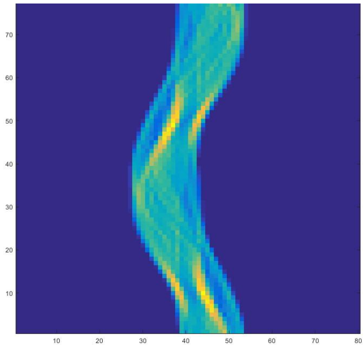


Figure: Activity function f_* (left) and attenuation function μ_* (right).

SPECT Data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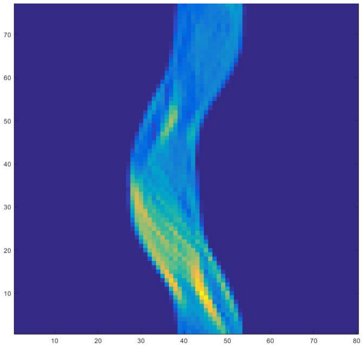
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$$A(f,\mu)(s,\omega) := \int_{\mathbb{R}} f(s\omega^{\perp} + t\omega) \exp\left(- \int_{t}^{\infty} \mu(s\omega^{\perp} + r\omega)\, dr \right)\, dt \,.$$

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$\lambda_k^\delta = 0$	3433	489 s
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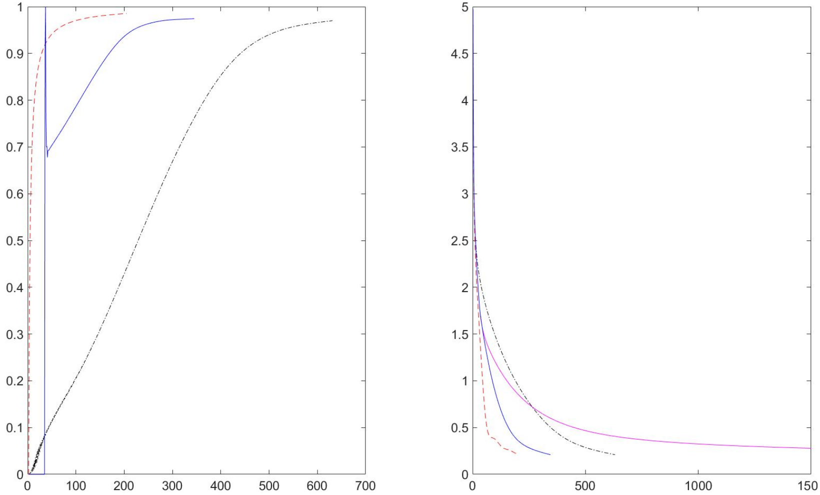
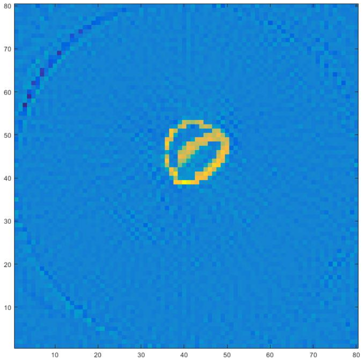
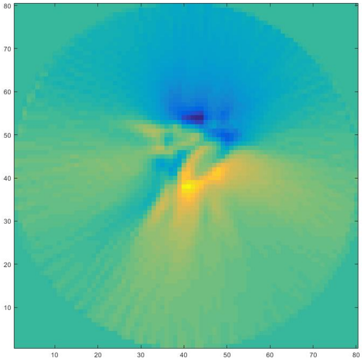
Evolution of λ_k^δ and Residuals



SPECT Reconstruction





Conclusion

Two-Point Gradient (TPG) methods

$$z_k^\delta = x_k^\delta + \lambda_k^\delta (x_k^\delta - x_{k-1}^\delta) \,,$$
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- converge under standard assumptions,
- are very easy to implement,
- require no more computation time than Landweber iteration,
- and lead to a considerable speed-up in practise.

Future Work

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- Convergence rates of the form

$$\left\| x_{k_*}^{\delta} - x^{\dagger} \right\| = \mathcal{O}\left(\delta^{\frac{2\mu}{2\mu+1}}\right), \qquad k_*(\delta, y^{\delta}) = \psi(\delta).$$

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- Nonlinear problems and original Nesterov parameter

$$\lambda_k^\delta = \frac{k-1}{k+\alpha-1} \,.$$

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- Nonlinear problems and original Nesterov parameter

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- Analysis only under local convexity assumption.
- Weakening of convexity assumption to, e.g., quasi-convexity.

References

- H. Attouch, J. Peypouquet, The rate of convergence of Nesterov's accelerated forward-backward method is actually $o(k^{-2})$, SIAM Journal on Optimization
- S. Hubmer, R. Ramlau Convergence Analysis of a Two-Point Gradient Method for Nonlinear Ill-Posed Problems, DK Preprint Series (2017)
- Y. Nesterov, A method of solving a convex programming problem with convergence rate $O(1/k^2)$, Soviet Mathematics Doklady, 27, 2, 1983
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Thank you for your attention!