Invariant algebraic curves of rational vector fields and their explicit rational solutions

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FWF
Der Wissenschaftsfonds.

JOHANNES KEPLER UNIVERSITÄT LINZ | JKU

ACA 2010
Vlora, Albania
June 24-27, 2010

Outline

Introduction to rational vector fields

Consider the autonomous rational system

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases} \tag{1}$$

where $M_1, N_1, M_2, N_2 \in \mathbb{K}[s,t]$, $M_1, M_2 \neq 0$, $\mathbb{K}$ is a field of constants. (e.g. $\mathbb{K} = \overline{\mathbb{Q}}$)

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- The system defines a vector field on the affine plane excluding two curves $M_1(s,t) = 0$ and $M_2(s,t) = 0$.
- We study a method for finding explicit rational solutions of a given degree, i.e., finding $(s(x), t(x)) \in \overline{\mathbb{K}}(x)^2$ satisfying the differential system.

Invariant algebraic curves of a rational vector field

Lemma

If $(s(x), t(x))$ is a non-trivial rational solution of the differential system (1), then the irreducible implicit equation $F(s,t) = 0$ of $(s(x), t(x))$ satisfies the relation

$$F_s \cdot N_1 M_2 + F_t \cdot N_2 M_1 = F \cdot K$$

for some bivariate polynomial K, where F_s and F_t are partial derivatives of F with respect to s and t.

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Of course, $F(s, t) = 0$ must be a rational algebraic curve.

Lemma

Let $F(s,t) = 0$ *be an irreducible algebraic curve such that*

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for some bivariate polynomial K*. Let* $(s(x), t(x))$ *be a rational parametrization of* $F(s,t) = 0$*. If* $M_1(s(x), t(x)) \neq 0$ *and* $M_2(s(x), t(x)) \neq 0$*, then*

$$s'(x) \cdot \frac{N_2(s(x), t(x))}{M_2(s(x), t(x))} - t'(x) \cdot \frac{N_1(s(x), t(x))}{M_1(s(x), t(x))} = 0.$$

Definition

An algebraic curve $F(s,t)=0$ is called an invariant algebraic curve of the rational vector field (1) iff

$$F_s \cdot N_1 M_2 + F_t \cdot N_2 M_1 = F \cdot K \tag{2}$$

for some bivariate polynomial K, $F \nmid M_1$ and $F \nmid M_2$.

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In addition, if $F(s, t) = 0$ is also a rational algebraic curve, then we call $F(s, t) = 0$ a rational invariant algebraic curve.

Main strategy

- Find the invariant algebraic curves, $F(s,t)=0$, of the rational differential system (1), i.e.,

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- Decide whether $F(s,t)=0$ gives a rational solution or not. In the affirmative case we compute a rational solution corresponding to the invariant algebraic curve $F(s,t)=0$.

Issues on finding invariant algebraic curves (briefly)

- Denote that $D := N_1 M_2 \cdot \frac{\partial}{\partial s} + N_2 M_1 \cdot \frac{\partial}{\partial t}$. Then D is a differential operator and one can define $F(s,t) = 0$ to be an invariant algebraic curve of D iff

$$DF = F \cdot K$$

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  for some polynomial K.
- The degree of the cofactor K is bounded by $\max\{\deg(N_1 M_2), \deg(N_2 M_1)\} - 1$.

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- Let $F_1(s,t)$ and $F_2(s,t)$ be irreducible polynomials. Then

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- As a corollary of Darboux's Theorem the degree of any irreducible invariant algebraic curves is bounded by some natural number N. However how to find such upper bound effectively is still an unsolved problem (Poincaré problem).

Example

Consider the polynomial system

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Let $d = 2$. The set of irreducible invariant algebraic curves of (3) of degree at most 2 is

$$\begin{aligned} \{&[t-1,\ 2s], \\ &[\alpha(t-1)+s,\ \alpha(t-1)+s], \\ &[-\alpha(t-1)+s,\ -\alpha(t-1)+s], \\ &[s^2+t^2+(-1-C)t+C,\ 2s]\} \end{aligned}$$

where $\alpha = RootOf(Z^2+1)$ and C is an arbitrary constant.

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where $\alpha = RootOf(Z^2+1)$ and C is an arbitrary constant.
In this example there is no irreducible invariant algebraic curves of degree higher than 2.

Finding rational solutions by parametrizing invariant algebraic curves

- Given an irreducible rational invariant algebraic curve $F(s,t) = 0$ of the system (1), i.e.,

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- Find rational parametrizations $(s(x), t(x))$ of $F(s,t)=0$ such that

$$\begin{cases} s'(x) = \dfrac{N_1(s(x),t(x))}{M_1(s(x),t(x))} \\ t'(x) = \dfrac{N_2(s(x),t(x))}{M_2(s(x),t(x))}. \end{cases}$$

Main theorem

Let $F(s,t) = 0$ be a rational algebraic curve such that

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$$T(x) = \frac{ax+b}{cx+d}$$

satisfying the autonomous differential equation

$$\begin{aligned} T' &= \frac{N_1(s(T), t(T))}{M_1(s(T), t(T)) \cdot s'(T)} \\ &= \frac{N_2(s(T), t(T))}{M_2(s(T), t(T)) \cdot t'(T)}. \end{aligned} \tag{4}$$

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In that case the solution is given by

$$\bar{s}(x)=s(T(x)), \quad \bar{t}(x)=t(T(x)).$$

Remarks

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- Every non-trivial rational solution of the rational differential system (1) forms a proper parametrization of a rational algebraic curve.
- The rational solvability of the autonomous differential equation (4) does not depend on the choice of a proper rational parametrization of $F(s,t)=0$.
- Let $(s(x), t(x))$ and $(\bar{s}(x), \bar{t}(x))$ be rational solutions of the differential system (1) corresponding to the same rational invariant algebraic curve $F(s,t)=0$. Then there exists a constant c such that

$$(s(x+c), t(x+c)) = (\bar{s}(x), \bar{t}(x)).$$

Example

Consider the rational system

$$\begin{cases} s' = \dfrac{-2(t-1)^2(s^2-(t-1)^2)}{((t-1)^2+s^2)^2} \\ t' = \dfrac{-4(t-1)^3 s}{((t-1)^2+s^2)^2}. \end{cases} \tag{5}$$

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In fact, it is enough to ask for irreducible invariant algebraic curves of the polynomial system

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They are

$$t-1=0, \alpha(t-1)+s=0, -\alpha(t-1)+s=0,$$
$$s^2+t^2+(-1-C)t+C=0,$$

where $\alpha = RootOf(Z^2+1)$, C is an arbitrary constant.

- Consider the line $t - 1 = 0$. It can be parametrized by

$$R = (x, 1).$$

Then we find a rational function $T(x)$ such that

$$T' = 0.$$

Thus it gives us a solution $\overline{s}(x) = C, \overline{t}(x) = 1$.

- Consider the rational invariant algebraic curve

$$F(s,t) = s^2 + t^2 + (-1 - C)t + C = 0.$$

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Hence

$$T(x) = \frac{C-1}{2x}.$$

Therefore, a rational solution corresponding to $F(s,t) = 0$ is

$$\bar{s}(x) = \frac{2(C-1)^2 x}{4x^2 + (C-1)^2}, \quad \bar{t}(x) = \frac{C((C-1)^2 + 4x^2}{4x^2 + (C-1)^2}.$$

The system

$$\begin{cases} s' = \dfrac{-2(t-1)^3(-(t-1)^2+s^2)}{((t-1)^2+s^2)^2} \\ t' = \dfrac{-4(t-1)^4 s}{((t-1)^2+s^2)^2}. \end{cases} \tag{7}$$

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has no rational solution different from the constant solutions $s(x) = C, t(x) = 1$. Because it has the same set of invariant algebraic curves and the autonomous differential equation for the transformation,

$$T' = \frac{-2T^4}{1+T^2},$$

has no rational solution.

Conclusion

1. We have provided a method for finding rational solutions of the differential system (1) using proper parametrizations of rational invariant algebraic curves.
2. We have proven that every rational solution of the differential system (1) forms a proper parametrization for its corresponding rational invariant algebraic curve.

Future works

1. We would like to study rational solutions of some special systems and differential equations.
   e.g.
   $$y' = R(x, y),$$
   where $R(x, y)$ is a rational function in x and y. This is equivalent to looking at the system
   $$\begin{cases} y' = R(x, y) \\ x' = 1. \end{cases}$$
2. Study a degree bound for rational solutions of the differential equation
   $$y' = R(x, y).$$

Thank you for your attention!