Rational Curves and Differential Equations

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Outline

Basics on rational algebraic curves

- An algebraic curve $F(s,t)=0$ is called a rational curve iff there is a non-constant rational mapping $\mathcal{P}(x)=(s(x),t(x))$ such that $F(s(x),t(x))=0$. Then $\mathcal{P}(x)$ is called a parametrization of $F(s,t)=0$.

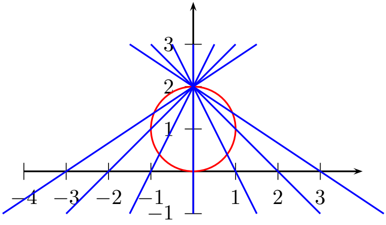
Example on rational curves:

The circle

$$s^2+(t-1)^2-1=0$$

can be parametrized by

$$\mathcal{P}(x)=\left(\frac{4x}{x^2+4},\frac{2x^2}{x^2+4}\right).$$



$$x\longmapsto \mathcal{P}(x)=(s(x),t(x)).$$

$$x\longmapsto T(x)\longmapsto \mathcal{P}(T(x))=(s(T(x)),t(T(x))).$$

Basics on rational algebraic curves

- The rational parametrizations of a rational curve $F(s,t)=0$ are not unique.
- A rational parametrization is called proper iff it is an invertible rational mapping and its inverse is also rational.
  In the example,

$$\mathcal{P}(x) = \left(\frac{4x}{x^2+4}, \frac{2x^2}{x^2+4}\right)$$

  is proper because its inverse is

$$x = \frac{2t}{s}.$$

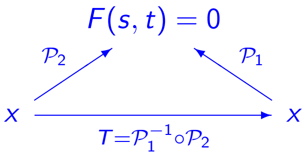
- A degree bound for proper parametrizations of a rational algebraic curve is given in [SW01].

Basics on rational algebraic curves

- If $\mathcal{P}_1(x) = (s_1(x), t_1(x))$ is a proper rational parametrization of $F(s,t) = 0$ and $\mathcal{P}_2(x) = (s_2(x), t_2(x))$ is any rational parametrization of $F(s,t) = 0$, then there exists a rational function $T(x)$ such that

$$\mathcal{P}_2(x) = \mathcal{P}_1(T(x)).$$

  Moreover, if $\mathcal{P}_2(x) = (s_2(x), t_2(x))$ is also proper, then $T(x) = \dfrac{ax+b}{cx+d}$ for some $a, b, c, d \in \overline{\mathbb{K}}$.



Basic notations in differential algebra

Let $\mathbb{K}(x)$ be the rational functions field over $\mathbb{K}$. Consider the usual derivation $\frac{d}{dx}$, denoted by $'$.

1. Differential rings: $\mathbb{K}(x)\{y\} := \mathbb{K}(x)[y, y', y'', \ldots]$.
2. Differential polynomials: elements in $\mathbb{K}(x)\{y\}$.
3. Differential ideals: $F \in \mathbb{K}(x)\{y\}$, $[F] :=< F, F', F'', \ldots >$.
4. Radical differential ideals: $\{F\} := \sqrt{[F]}$.
5. Initials: Let n be the order of F. Then the coefficient of F with respect to $y^{(n)}$ is called the initial of F.
6. Separants: Let n be the order of F. Then $\frac{\partial F}{\partial y^{(n)}}$ is called the separant of F.
7. Quotient ideals: $\{F\} : S := \{A \in \mathbb{K}(x)\{y\} \mid AS \in \{F\}\}$.

Lemma

Let F be an irreducible differential polynomial. The differential ideal $(\{F\} : S)$ is a prime differential ideal and

$$\{F\} = (\{F\} : S) \cap \{F, S\}.$$

General solutions of algebraic ODEs

Definition

A generic zero of $(\{F\} : S)$ is called a general solution of $F = 0$.

Example:

Let $F = y'^2 - 4y$. Then $S = 2y'$; $F' = 2y'y'' - 4y' = 2y'(y'' - 2)$.

$$\{F\} : S = \{y'^2 - 4y, y'' - 2\}, \quad \{F, S\} = \{y'^2 - 4y, y'\}.$$

$$\mathcal{Z}(\{F\} : S) = \{y = (x + c)^2 \mid c \text{ is any constant}\}.$$

$$\mathcal{Z}(\{F, S\}) = \{y = 0\}.$$

Definition

A rational general solution of $F = 0$ is a general solution of $F = 0$ of the form

$$y = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0},$$

where a_i, b_j are constants.

Autonomous algebraic ODEs (Feng and Gao, [FG04])

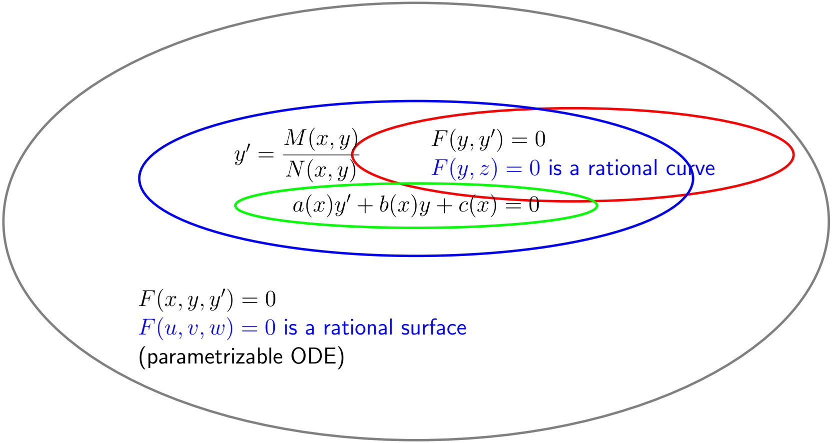
$$F(y, y') = 0, \quad F \in \mathbb{K}[y, z], \quad \mathrm{char}(\mathbb{K}) = 0$$

where $\mathbb{K}$ is a field of constants, y is an indeterminate over $\mathbb{K}(x)$.

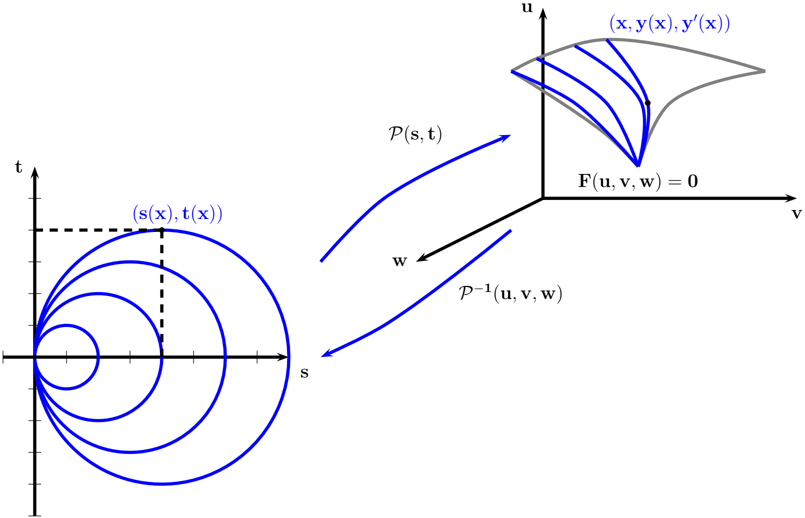
- **Key observation:**

Non-trivial rational solutions	Proper parametrizations
$F(y, y') = 0$	$F(y, z) = 0$
$y = f(x)$	$(f(x), f'(x))$
y=g(T(x))	(g(x), h(x))
Rational general solution is y=g(T(x+C)) for arbitrary constant C.	Find $T(x) = \dfrac{ax + b}{cx + d}$ with g'(T).T'=h(T)

GOAL- Parametrizable non-autonomous ODEs of order 1, $F(x, y, y') = 0$, $F \in \mathbb{K}[u, v, w]$.



Non-autonomous case $F(x, y, y') = 0$



Non-autonomous case $F(x, y, y') = 0$

- We assume that $F(x, y, z) = 0$ has a proper parametrization

$$\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)).$$

Then we associate to $F(x, y, y') = 0$ an autonomous differential system

$$\begin{cases} s' = \dfrac{f_1(s,t)}{g(s,t)} \\ t' = \dfrac{f_2(s,t)}{g(s,t)}, \end{cases} \tag{1}$$

which is called the associated system of $F(x, y, y') = 0$ with respect to $\mathcal{P}(s, t)$, where

$$\begin{aligned} f_1(s,t) :=& \frac{\partial \chi_2(s,t)}{\partial t} - \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial t}, \\ f_2(s,t) :=& \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial s} - \frac{\partial \chi_2(s,t)}{\partial s}, \\ g(s,t) :=& \frac{\partial \chi_1(s,t)}{\partial s} \cdot \frac{\partial \chi_2(s,t)}{\partial t} - \frac{\partial \chi_1(s,t)}{\partial t} \cdot \frac{\partial \chi_2(s,t)}{\partial s}. \end{aligned} \tag{2}$$

- Note that the associated system (1) is constructed in such a way that if $(s(x), t(x))$ is a rational solution of the system, then $\mathcal{P}(s(x), t(x))$ has the form

$$\mathcal{P}(s(x), t(x)) = (x + C, \varphi(x), \varphi'(x))$$

for some constant C. Therefore,

$$y = \chi_2(s(x - C), t(x - C))$$

is a rational solution of $F(x, y, y') = 0$.

Rational general solutions

Theorem ([NW10])

There is a one-to-one correspondence between rational general solutions of the algebraic differential equation $F(x, y, y') = 0$, which is parametrized by $\mathcal{P}(s, t)$, and rational general solutions of its associated system with respect to $\mathcal{P}(s, t)$.

The associated systems of autonomous ODEs, $F(y, y') = 0$.

(A recovery of Feng and Gao's result)

$$\boxed{F(y, y') = 0,\ F(f(t), g(t)) = 0,\ \mathcal{P}(s, t) = (s, f(t), g(t))}$$

The associated system w.r.t. $\mathcal{P}(s, t)$ is

$$\begin{cases} s' = 1 \\ t' = \dfrac{g(t)}{f'(t)}. \end{cases}$$

A rational general solution of the associated system is of the form

$$s(x) = x + C, \quad t(x) = \frac{ax + b}{cx + d},$$

where C is an arbitrary constant. A rational general solution of $F(y, y') = 0$ is

$$y = f(t(x - C)).$$

The autonomous system of ODEs of order 1

The associated system of a parametrizable ODE, $F(x, y, y') = 0$, has the form

$$\begin{cases} s' = \dfrac{M_1(s,t)}{N_1(s,t)} \\ t' = \dfrac{M_2(s,t)}{N_2(s,t)} \end{cases} \qquad (3)$$

where $M_1, N_1, M_2, N_2 \in \mathbb{K}[s,t]$, $N_1, N_2 \neq 0$, $\gcd(M_1, N_1) = 1$, $\gcd(M_2, N_2) = 1$ and $\mathbb{K}$ is a field of constants (e.g. $\mathbb{K} = \overline{\mathbb{Q}}$).

1. Find the implicit equations $G(s,t) = 0$ of the parametric solution curves $(s(x), t(x))$.
2. Parametrize $G(s,t) = 0$ to obtain a rational solution.

Rational invariant algebraic curve

Lemma

Every non-trivial rational solution of the associated system (3) *is corresponding to a rational algebraic curve, whose implicit equation* $G(s,t) = 0$ *is satisfying the relation*

$$G_s \cdot M_1 N_2 + G_t \cdot M_2 N_1 = G \cdot K$$

for some polynomial K.

Definition

A rational algebraic curve $G(s,t) = 0$ is called a rational invariant algebraic curve of the system (3) iff

$$G_s \cdot M_1 N_2 + G_t \cdot M_2 N_1 = G \cdot K$$

for some polynomial K.

Rational invariant algebraic curve

Lemma

Let $G(s,t)=0$ be a rational invariant algebraic curve of the system. If $(s(x),t(x))$ is a rational parametrization of $G(s,t)=0$ with $N_1(s(x),t(x))\neq 0$ and $N_2(s(x),t(x))\neq 0$, then

$$s'(x)\cdot\frac{M_2(s(x),t(x))}{N_2(s(x),t(x))}=t'(x)\cdot\frac{M_1(s(x),t(x))}{N_1(s(x),t(x))}.$$

- Recall that we look for the rational solutions of the system

$$\begin{cases} s'=\dfrac{M_1(s,t)}{N_1(s,t)} \\ t'=\dfrac{M_2(s,t)}{N_2(s,t)} \end{cases}$$

Definition

A rational invariant algebraic curve $G(s, t) = 0$ of the system (3) is called a rational solution curve iff there is a parametrization of the curve $G(s, t) = 0$ satisfying the system (3).

Reparametrization Theorem

Theorem ([Ngo10])

Let $G(s,t)=0$ be a rational invariant algebraic curve of the system (3) such that $G \nmid N_1$ and $G \nmid N_2$. Let $(s(x), t(x))$ be a proper rational parametrization of $G(s,t)=0$. Then $G(s,t)=0$ is a rational solution curve of the system (3) if and only if one of the following differential equations has a rational solution $T(x)$.

1. $T' = \frac{1}{s'(T)} \cdot \frac{M_1(s(T), t(T))}{N_1(s(T), t(T))}$ *when* $s'(x) \not\equiv 0$.
2. $T' = \frac{1}{t'(T)} \cdot \frac{M_2(s(T), t(T))}{N_2(s(T), t(T))}$ *when* $t'(x) \not\equiv 0$.
3. *When $s'(x) \not\equiv 0$ and $t'(x) \not\equiv 0$, we can choose either of two differential equations (because of the lemma above).*

If there is a rational solution for $T(x)$, then the rational solution of the system (3) corresponding to $G(s,t)=0$ is

$$(s(T(x)), t(T(x))).$$

Facts [Ngo10]

1. The rational solutions of

$$T' = \frac{1}{s'(T)} \cdot \frac{M_1(s(T), t(T))}{N_1(s(T), t(T))} \quad \text{when } s'(x) \not\equiv 0$$

   or

$$T' = \frac{1}{t'(T)} \cdot \frac{M_2(s(T), t(T))}{N_2(s(T), t(T))} \quad \text{when } t'(x) \not\equiv 0$$

   must be linear rational functions, i.e.,

$$T(x) = \frac{ax + b}{cx + d},$$

   where a, b, c and d are constants.
2. The theorem does not depend on the choice of a proper rational parametrization $(s(x), t(x))$ of $G(s, t) = 0$.
3. Every rational solution of the system (3) forms a proper rational parametrization of a rational algebraic curve.

How to find an invariant algebraic curve?

- Need a degree bound (Poincaré problem).
- When we have a degree bound, using undetermine coefficient method to find $G(s,t)$ with

$$G_s \cdot M_1 N_2 + G_t \cdot M_2 N_1 = G \cdot K.$$

Example for invariant algebraic curves but not solution curves

Consider the system

$$\begin{cases} s' = \dfrac{-2(t-1)^3(-(t-1)^2+s^2)}{((t-1)^2+s^2)^2} \\ t' = \dfrac{-4(t-1)^4 s}{((t-1)^2+s^2)^2}. \end{cases} \qquad (4)$$

The set of invariant algebaric curves is

$$\{t-1=0, \; s^2+t^2+(-1-C)t+C=0\}.$$

1. The line $t-1=0$ is a rational solution curve, corresponding to the rational solution $s(x)=c, t(x)=1$.
2. The family of curves

$$s^2+t^2+(-1-C)t+C=0$$

are not rational solution curves.

Consider the rational invariant algebraic curve

$$s^2 + t^2 + (-1 - C)t + C = 0.$$

A proper rational parametrization is

$$R = \left(\frac{(C-1)x}{1+x^2}, \frac{Cx^2+1}{1+x^2} \right).$$

The equation for reparametrization

$$T' = \frac{-2T^4}{1+T^2}$$

has NO rational solution. Therefore,

$$s^2 + t^2 + (-1 - C)t + C = 0$$

is not a rational solution curve.

Example 1

Consider the linear differential equation

$$F(x, y, y') \equiv -(1+2x)y' + 2y - 24x^2 - 32x^3 = 0.$$

The algebraic surface $F(x, y, z) = 0$ has a proper parametrization

$$\mathcal{P}(s, t) = \left(s - \frac{t}{2},\ 4s^2 - t^2 + 2t,\ 16st - 4t^2 - 16s^2 + 4t \right).$$

The associated system with respect to $\mathcal{P}(s, t)$ is

$$\begin{cases} s' = -2s + t + 1 \\ t' = -4s + 2t. \end{cases}$$

- The family of irreducible invariant algebraic curves is

$$4s^2 - 4st + t^2 + 2t - C = 0.$$

- At infinity the system is given by

$$\begin{cases} u' = -(u-2)^2 - (u-2)v - 2v \\ v' = -v(u-2+v). \end{cases}$$

It has a unique singularity at $(2,0)$. In fact, this is a non-dicritical singularity. Since the degree of the system is 1, by [Car94], the degree of any irreducible invariant algebraic curve of the system is bounded by 1+2=3.

A proper parametrization of the family

$$4s^2 - 4st + t^2 + 2t - C = 0$$

is

$$\left(-\frac{1}{2}\cdot\frac{4\sqrt{C}x^2+4x-\sqrt{C}}{4x^2-4x+1}, -2\cdot\frac{2\sqrt{C}x+1-\sqrt{C}}{4x^2-4x+1}\right).$$

The differential equation for the reparametrization is

$$T' = 2\left(T-\frac{1}{2}\right)^2.$$

Thus

$$T(x) = \frac{x-1}{2x}.$$

It follows that the rational general solution of the associated system is

$$s(x) = -x^2 + (1+\sqrt{C})x - \frac{1}{2}\sqrt{C},\ t(x) = 2\left(-x+\sqrt{C}\right)x.$$

We have

$$\chi_1(s(x), t(x)) = x - \frac{1}{2}\sqrt{C}.$$

Substituting $s = s(x + \frac{1}{2}\sqrt{C})$ and $t = t(x + \frac{1}{2}\sqrt{C})$ into $\chi_2(s, t)$ we get the rational general solution of the differential equation $F(x, y, y') = 0$,

$$y(x) = -8x^3 + 2Cx + C.$$

Example 2

Consider the differential equation

$$F(x, y, y') \equiv y'^3 - 4xyy' + 8y^2 = 0.$$

The algebraic surface $F(x, y, z) = 0$ has a proper parametrization

$$\mathcal{P}(s, t) = (t, -4s^2 \cdot (2s - t), -4s \cdot (2s - t)).$$

The associated system with respect to $\mathcal{P}(s, t)$ is

$$\begin{cases} s' = \dfrac{1}{2} \\ t' = 1. \end{cases}$$

Similarly, we can solve the system to obtain

$$s(x) = \frac{x}{2},\ t(x) = x - C$$

where C is an arbitrary constant. Therefore, the rational general solution of the differential equation $F(x, y, y') = 0$ is

$$y(x) = -C(x + C)^2.$$

Some solvable classes

(joint work with J. Rafael Sendra)

Theorem

The following classes are solvable via their associated systems.

- $F(y, y') = 0$, $F(y, z) = 0$ *is a rational curve;*
- $F\left(y - \dfrac{\beta(x-\gamma)}{\alpha}, y'\right) = 0$, $F(y, z) = 0$ *is a rational curve;*
- $F\left(x - \dfrac{\alpha(y-\gamma)}{\beta}, y'\right) = 0$, $F(y, z) = 0$ *is a rational curve;*

  where α, β *and* γ *are constants.*

Some solvable classes

$$F\left(y-\frac{\beta(x-\gamma)}{\alpha},y'\right)=0$$

$$F(f(s),g(s))=0$$

$$\mathcal{P}(s,t)=\begin{pmatrix}\gamma\\ f(s)\\ g(s)\end{pmatrix}+t\cdot\begin{pmatrix}\alpha\\ \beta\\ 0\end{pmatrix}$$

$$\begin{cases} s'=\dfrac{\beta-\alpha g(s)}{-\alpha f'(s)}\\ t'=\dfrac{1}{\alpha}\end{cases}$$

$$F\left(x-\frac{\alpha(y-\gamma)}{\beta},y'\right)=0$$

$$F(f(s),g(s))=0$$

$$\mathcal{P}(s,t)=\begin{pmatrix}f(s)\\ \gamma\\ g(s)\end{pmatrix}+t\cdot\begin{pmatrix}\alpha\\ \beta\\ 0\end{pmatrix}$$

$$\begin{cases} s'=\dfrac{\beta-\alpha g(s)}{\beta f'(s)}\\ t'=\dfrac{g(s)}{\beta}\end{cases}$$

Some solvable classes

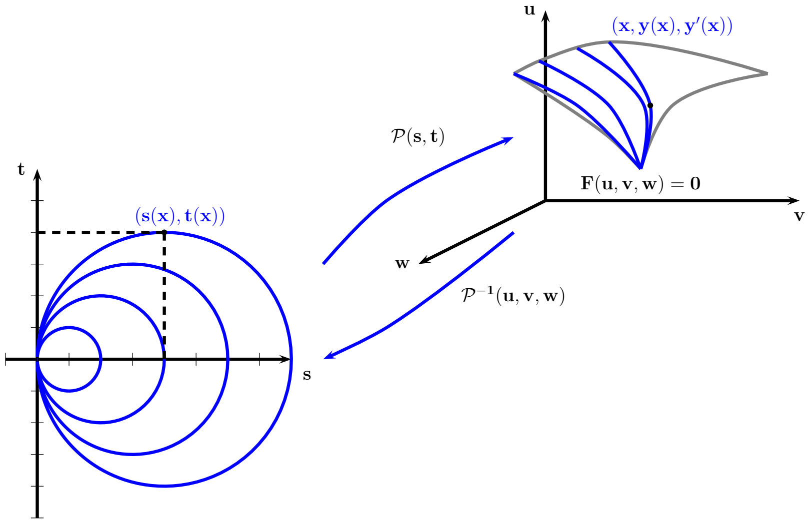
Theorem

Let $F(x, y, y') = 0$ *be a parametrizable ODE of order 1. Then*

$$F(x, y, y') = 0 \;\; \text{and} \;\; F(x, bx + ay, b + ay') = 0$$

have the same associated systems, where $a, b \in \mathbb{K}$ *and* $a \neq 0$*. In particular, if* $F(x, y, y') = 0$ *is solvable, then* $F(x, bx + ay, b + ay') = 0$ *is solvable.*

Conclusion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Thank you for your attention!

!Gracias por su atención!

Cảm ơn sự chú ý của bạn!