RATIONAL CURVES AND DIFFERENTIAL EQUATIONS

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Abstract

We study the rational general solutions of a non-autonomous algebraic ordinary differential equation (ODE) of order 1. The geometric approach of R. Feng and X-S. Gao ([FG04], [FG06]) in the autonomous case can be extended to the non-autonomous ODEs of order 1 in a natural way provided a proper rational parametrization of the corresponding algebraic surface.

The work leads to studying a system of autonomous ODEs of order 1 and of degree 1. We call it the associated system with respect to a parametrization of the original ODE. If we can solve this associated system for its rational general solutions, then we shall obtain the rational general solutions of the original ODE by using the parametrization map.

Autonomous ODEs

The following algorithm can be found in [FG04].

Algorithm 1.

Input: $F(y, y') = 0, F \in \overline{\mathbb{Q}}[y, z]$.

Output: A rational general solution of $F(y, y') = 0$ if any.

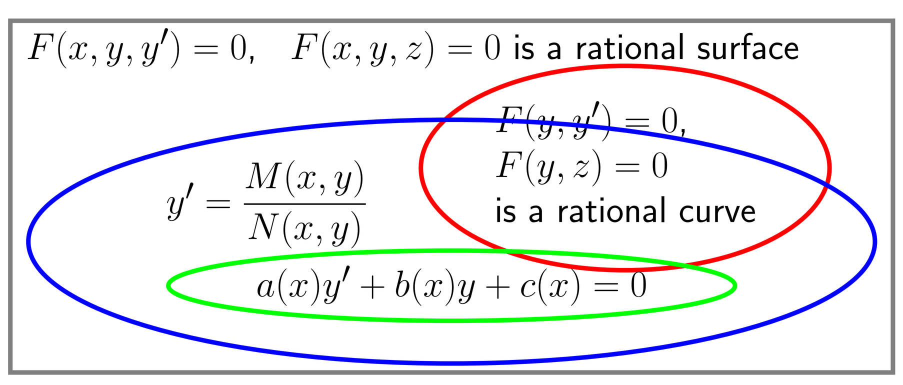
1. if $F(y, z) = 0$ is not a rational curve, then return "$F(y, y') = 0$ has no rational solution";
2. else compute a proper rational parametrization of $F(y, z) = 0$, say $(f(t), g(t))$.
3. compute a rational function $T(x) = \dfrac{ax+b}{cx+d}$ such that
$$f(T(x))' = g(T(x)).$$
4. if there is no such $T(x)$, then return "$F(y, y') = 0$ has no rational solution";
5. else return the rational general solution
$$y = f(T(x + C)),$$
where C is an arbitrary constant.

Goal

We extend the above geometric approach to parametrizable non-autonomous ODEs. We assume that the non-autonomous ODE $F(x, y, y') = 0$ is parametrizable, i.e. the corresponding algebraic surface $F(x, y, z) = 0$ is a rational surface with a proper rational parametrization

$$\mathcal{P}(s, t) = (\chi_1(s, t), \chi_2(s, t), \chi_3(s, t)),$$

where the coefficients of χ_i belong to some field of constants containing the ground field of F. This is a naturally extended class of the class of autonomous ODEs of order 1 with rational solutions.



Associated system

This leads to studying the following system of autonomous ODEs of order 1 and of degree 1 in the derivatives, which is called the associated system with respect to $\mathcal{P}(s, t)$,

$$\begin{cases} s' = \dfrac{f_1(s, t)}{g(s, t)} \\ t' = \dfrac{f_2(s, t)}{g(s, t)}, \end{cases} \tag{1}$$

where $f_1(s, t), f_2(s, t), g(s, t)$ are rational functions in s, t and defined by

$$\begin{aligned} f_1(s, t) &= \frac{\partial \chi_2(s, t)}{\partial t} - \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial t}, \\ f_2(s, t) &= \chi_3(s, t) \cdot \frac{\partial \chi_1(s, t)}{\partial s} - \frac{\partial \chi_2(s, t)}{\partial s}, \\ g(s, t) &= \frac{\partial \chi_1(s, t)}{\partial s} \cdot \frac{\partial \chi_2(s, t)}{\partial t} - \frac{\partial \chi_1(s, t)}{\partial t} \cdot \frac{\partial \chi_2(s, t)}{\partial s}. \end{aligned} \tag{2}$$

Note that the associated system is constructed in such a way that if $(s(x), t(x))$ is a rational solution of the system, then we have

$$\chi_1(s(x), t(x)) = x + c,$$

for some constant c and

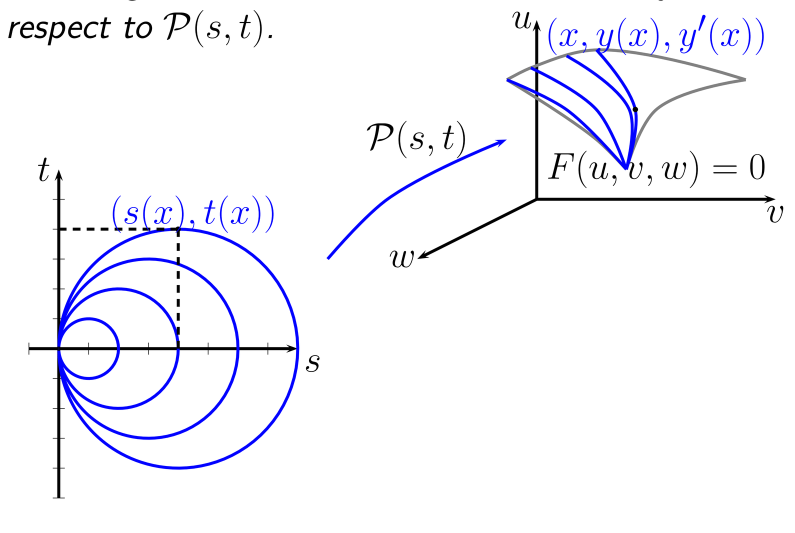
$$\chi_2(s(x - c), t(x - c))' = \chi_3(s(x - c), t(x - c)).$$

Therefore,

$$y = \chi_2(s(x - c), t(x - c))$$

is a rational solution of $F(x, y, y') = 0$.

Theorem 1. *There is a one-to-one correspondence between rational general solutions of the parametrizable ODE $F(x, y, y') = 0$, which is parametrized by $\mathcal{P}(s, t)$, and rational general solutions of its associated system with respect to $\mathcal{P}(s, t)$.*



Special cases

We demonstrate the associated system of some special differential equations. Depending on how we parametrize the corresponding rational surface, the associated systems are looking differently.

$$F(y, y') = 0$$
$$F(s, f(t), g(t)) = 0$$
$$\begin{cases} s' = 1 \\ t' = \dfrac{g(t)}{f'(t)} \end{cases}$$

$F \equiv a(x)y' + b(x)y + c(x) = 0$	$F \equiv N(x, y)y' - M(x, y) = 0$
$F\left(s, t, \dfrac{-b(s)t - c(s)}{a(s)}\right) = 0$	$F\left(s, t, \dfrac{M(s, t)}{N(s, t)}\right) = 0$
$\begin{cases} s' = 1 \\ t' = \dfrac{-b(s)t - c(s)}{a(s)} \end{cases}$	$\begin{cases} s' = 1 \\ t' = \dfrac{M(s, t)}{N(s, t)} \end{cases}$

Invariant algebraic curves

Consider the system of autonomous ODEs

$$\begin{cases} s' = \dfrac{M_1(s, t)}{N_1(s, t)} \\ t' = \dfrac{M_2(s, t)}{N_2(s, t)}, \end{cases} \tag{3}$$

where M_1, M_2, N_1 and N_2 are polynomials over some field of constants $\mathbb{K}$ and they are in the reduced forms.

Definition 1. An invariant algebraic curve of the system (3) is an algebraic curve $G(s, t) = 0$ such that

$$G_s M_1 N_2 + G_t M_2 N_1 = GK,$$

where G_s and G_t are partial derivatives of G with respect to s and t and K is some polynomial.

A rational invariant algebraic curve of the system (3) is an invariant algebraic curve and it is a rational curve.

Definition 2. A general invariant algebraic curve of the system (3) is an irreducible invariant algebraic curve $G(s, t) = 0$ with the property that for any $H \in \mathbb{K}[s, t]$, H is divisible by G if and only if $H = 0$.

- The set of invariant algebraic curves of the system (3) is the same as the one of the polynomial system
$$\begin{cases} s' = M_1(s, t) N_2(s, t) \\ t' = M_2(s, t) N_1(s, t). \end{cases}$$
- The implicit equation of any non-trivial rational solution of the system (3) defines a rational invariant algebraic curve of the system.

Algorithm 2.

Input: $G(s, t) = 0$ is a rational invariant algebraic curve of the system (3).

Output: A rational solution of the system (3) corresponding to the curve $G(s, t) = 0$ if any.

1. take an arbitrary proper rational parametrization of the curve $G(s, t) = 0$, say $(s(x), t(x))$.
2. find a linear rational function $T(x) = \dfrac{ax+b}{cx+d}$ such that
$$T' = \begin{cases} \dfrac{1}{s'(T)} \cdot \dfrac{M_1(s(T), t(T))}{N_1(s(T), t(T))} & \text{if } s'(x) \neq 0 \\ \dfrac{1}{t'(T)} \cdot \dfrac{M_2(s(T), t(T))}{N_2(s(T), t(T))} & \text{if } t'(x) \neq 0. \end{cases} \tag{4}$$
3. if there is no such $T(x)$, then return "No rational solution corresponding to the curve $G(s, t) = 0$";
4. else return
$$(s(T(x)),\ t(T(x))).$$

Singularities

Consider the polynomial differential system

$$\begin{cases} s' = P(s, t) \\ t' = Q(s, t). \end{cases} \tag{5}$$

- The singularities of the system are the common intersection points of $P(s, t) = 0$ and $Q(s, t) = 0$ in the projective plane.
- The singularities of an irreducible invariant algebraic curve of the polynomial system are among the singularities of the system.
- The analysis of singularities of the system gives a degree bound for irreducible invariant algebraic curves in terms of the degree of the polynomial system in the non-dicritical case, where every singularities of the system is non-dicritical ([Car94]).
- The irreducible invariant algebraic curves of the system can be found by using a degree bound and undetermined coefficients.
- If the polynomial system is a linear system, then the degree of any irreducible rational invariant algebraic curve is at most 4 ([NW10]).

Algorithm in non-dicritical case

Algorithm 3.

Input: $F(x, y, y') = 0$, $\mathcal{P}(s, t) = (\chi_1, \chi_2, \chi_3)$ a proper rational parametrization of the surface $F(x, y, z) = 0$.

Output: A rational general solution of $F(x, y, y') = 0$ in the non-dicritical case if any.

1. compute the associated system with respect to $\mathcal{P}(s, t)$;
2. compute the set of singularities;
3. if there is a dicritical singularity, then return "We need a degree bound";
4. else find a set of rational invariant algebraic curves with the degree bound;
5. if there is no rational general invariant algebraic curve, then return "There is no rational general solution";
6. else use the Algorithm 2 to find a rational solution of the associated system corresponding to the rational general invariant algebraic curve, say $(s(x), t(x))$;
7. compute the constant $c = \chi_1(s(x), t(x)) - x$;
8. return $y(x) = \chi_2(s(x - c), t(x - c))$.

Example 1

Consider the linear differential equation

$$F(x, y, y') \equiv -(1 + 2x)y' + 2y - 24x^2 - 32x^3 = 0.$$

The algebraic surface $F(x, y, z) = 0$ has a proper parametrization

$$\mathcal{P} = \left(s - \frac{t}{2},\ 4s^2 - t^2 + 2t,\ 16st - 4t^2 - 16s^2 + 4t\right).$$

The associated system with respect to $\mathcal{P}(s, t)$ is

$$\begin{cases} s' = -2s + t + 1 \\ t' = -4s + 2t. \end{cases}$$

At infinity the system is given by

$$\begin{cases} u' = -(u - 2)^2 - (u - 2)v - 2v \\ v' = -v(u - 2 + v). \end{cases}$$

It has a unique singularity $(2, 0)$ at infinity. In fact, this is a non-dicritical singularity. Since the degree of the system is 1, by ([Car94]), the degree of any irreducible invariant algebraic curve of the system is bounded by 1+2=3. In this example the rational general invariant algebraic curve is

$$4s^2 - 4st + t^2 + 2t - C = 0,$$

where C is an arbitrary constant. A proper rational parametrization of this curve is

$$\left(-\frac{1}{2} \cdot \frac{4\sqrt{C}x^2 + 4x - \sqrt{C}}{4x^2 - 4x + 1},\ -2 \cdot \frac{2\sqrt{C}x + 1 - \sqrt{C}}{4x^2 - 4x + 1}\right).$$

The differential equation for the reparametrization is

$$T' = 2\left(T - \frac{1}{2}\right)^2.$$

Thus

$$T(x) = \frac{x - 1}{2x}.$$

It follows that a rational general solution of the associated system is

$$\begin{cases} s(x) = -x^2 + (1 + \sqrt{C})x - \frac{1}{2}\sqrt{C} \\ t(x) = 2\left(-x + \sqrt{C}\right)x. \end{cases}$$

We have

$$\chi_1(s(x), t(x)) = x - \frac{1}{2}\sqrt{C}.$$

Substituting $s = s(x + \frac{1}{2}\sqrt{C})$ and $t = t(x + \frac{1}{2}\sqrt{C})$ into $\chi_2(s, t)$ we obtain a rational general solution of the differential equation $F(x, y, y') = 0$,

$$y(x) = -8x^3 + 2Cx + C.$$

Example 2

Consider the non-linear differential equation

$$F(x, y, y') \equiv y'^3 - 4xyy' + 8y^2 = 0.$$

The algebraic surface $F(x, y, z) = 0$ has a proper parametrization

$$\mathcal{P}(s, t) = (t, -4s^2 \cdot (2s - t), -4s \cdot (2s - t)).$$

The associated system with respect to $\mathcal{P}(s, t)$ is

$$\begin{cases} s' = \dfrac{1}{2} \\ t' = 1. \end{cases}$$

Similarly, we can solve the system to obtain

$$s(x) = \frac{x}{2},\ t(x) = x - C$$

where C is an arbitrary constant. Therefore, the rational general solution of the differential equation $F(x, y, y') = 0$ is

$$y(x) = -C(x + C)^2.$$

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