# Rational general solutions of first order non-autonomous parametric ODEs

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# Outline

# Introduction

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- If $(r(x), s(x))$ is a proper rational parametrization of $F(y, z) = 0$, then under certain "differential compatibility conditions" one obtains a rational general solution of $F(y, y') = 0$ from $r(x)$.

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- A rational solution $y = f(x)$ defines a rational space curve

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  on the surface $F(x, y, z) = 0$.
- Assume in addition that the surface $F(x, y, z) = 0$ is parametrized by a proper rational parametrization $\mathcal{P}(s, t)$. We will find the "differential compatibility conditions" on the coordinate functions of $\mathcal{P}(s, t)$.

# Construction of solutions

Let

$$\mathcal{P}(s,t) = (\chi_1(s,t), \chi_2(s,t), \chi_3(s,t))$$

be a proper parametrization of $F(x,y,z) = 0$, where

$$\chi_1(s,t), \chi_2(s,t), \chi_3(s,t) \in \overline{\mathbb{Q}}(s,t).$$

Suppose that the inverse of $\mathcal{P}(s,t)$ is

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$$\mathcal{P}^{-1}(x, f(x), f'(x)) = (s(x), t(x)),$$

which defines a rational plane curve and satisfies the relation

$$\begin{cases} \chi_1(s(x), t(x)) = x \\ \chi_2(s(x), t(x)) = f(x) \\ \chi_3(s(x), t(x)) = f'(x). \end{cases}$$

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$$\begin{cases} \dfrac{\partial \chi_1(s(x), t(x))}{\partial s} \, s'(x) + \dfrac{\partial \chi_1(s(x), t(x))}{\partial t} \, t'(x) = 1 \\ \\ \dfrac{\partial \chi_2(s(x), t(x))}{\partial s} \, s'(x) + \dfrac{\partial \chi_2(s(x), t(x))}{\partial t} \, t'(x) = \chi_3(s(x), t(x)) \end{cases} \quad (2)$$

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$\Downarrow$ $\exists c$ constant

$$\begin{cases} \chi_1(s(x - c), t(x - c)) = x \\ [\chi_2(s(x - c), t(x - c))]' = \chi_3(s(x - c), t(x - c)) \end{cases} \quad (3)$$

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$y = \chi_2(s(x-c), t(x-c))$ is a rational solution of $F(x, y, y') = 0$.

Consider the linear system (2)

$$\begin{cases} \dfrac{\partial \chi_1(s(x), t(x))}{\partial s} \, s'(x) + \dfrac{\partial \chi_1(s(x), t(x))}{\partial t} \, t'(x) = 1 \\[2ex] \dfrac{\partial \chi_2(s(x), t(x))}{\partial s} \, s'(x) + \dfrac{\partial \chi_2(s(x), t(x))}{\partial t} \, t'(x) = \chi_3(s(x), t(x)). \end{cases}$$

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Let

$$\begin{aligned} g(s,t) &:= \frac{\partial \chi_1(s,t)}{\partial s} \cdot \frac{\partial \chi_2(s,t)}{\partial t} - \frac{\partial \chi_1(s,t)}{\partial t} \cdot \frac{\partial \chi_2(s,t)}{\partial s}, \\ f_1(s,t) &:= \frac{\partial \chi_2(s,t)}{\partial t} - \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial t}, \\ f_2(s,t) &:= \frac{\partial \chi_2(s,t)}{\partial s} - \chi_3(s,t) \cdot \frac{\partial \chi_1(s,t)}{\partial s}. \end{aligned} \tag{4}$$

There are two cases

$$\text{either} \quad \begin{cases} g(s(x), t(x)) = 0 \\ f_1(s(x), t(x)) = 0 \end{cases} \qquad \text{or} \qquad \begin{cases} s'(x) = \dfrac{f_1(s(x), t(x))}{g(s(x), t(x))} \\ \\ t'(x) = -\dfrac{f_2(s(x), t(x))}{g(s(x), t(x))}. \end{cases} \tag{5}$$

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The second system is called the associated system of the equation $F(x, y, y') = 0$ with respect to $\mathcal{P}(s, t)$.

# Differential algebra notions

- $\overline{\mathbb{Q}}(x)$ the differential field of rational functions in $x$ with usual derivation $'$.
- $y$ an indeterminate over $\overline{\mathbb{Q}}(x)$.
- $\overline{\mathbb{Q}}(x)\{y\}$ the differential ring over $\overline{\mathbb{Q}}(x)$.
- Initial, separant of $F \in \overline{\mathbb{Q}}(x)\{y\}$ denoted by $I$ and $S$ respectively.

For any $G \in \overline{\mathbb{Q}}(x)\{y\}$ we have a unique representation

$$I^m S^n G = Q_k F^{(k)} + Q_{k-1} F^{(k-1)} + \cdots + Q_1 F' + Q_0 F + R$$

where

- $I$ is the initial of $F$, $S$ is the separant of $F$,
- $m, n, k \in \mathbb{N}$,
- $F^{(i)}$ is the $i$-th derivative of $F$,
- $Q_i, R \in \overline{\mathbb{Q}}(x)\{y\}$, $R$ is reduced with respect to $F$.

The $R$ is called the differential pseudo remainder of $G$ with respect to $F$, denoted by

$$\text{sprem}(G, F).$$

## Definition

A rational solution

$$\bar{y} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}$$

of $F(x, y, y') = 0$ is called a rational general solution if for any differential polynomial $G \in \overline{\mathbb{Q}}(x)\{y\}$ we have

$$G(\bar{y}) = 0 \iff \text{sprem}(G, F) = 0.$$

## Definition

Let $N_1(s,t), M_1(s,t), N_2(s,t), M_2(s,t) \in \overline{\mathbb{Q}}[s,t]$. A rational solution $(s(x), t(x))$ of the autonomous system

$$\begin{cases} s' = \dfrac{N_1(s,t)}{M_1(s,t)} \\ t' = \dfrac{N_2(s,t)}{M_2(s,t)} \end{cases}$$

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$$G(s(x), t(x)) = 0 \iff \text{sprem}(G, [M_1 s' - N_1, M_2 t' - N_2]) = 0.$$

**Lemma**

*If* $(s(x), t(x))$ *is a rational general solution of the associated system* (5) *and* $G \in \overline{\mathbb{Q}}(x)[s, t]$*, then*

$$G(s(x), t(x)) = 0 \Longleftrightarrow G = 0.$$

**Theorem**

*If the associated system (5) has a rational general solution, then there exists a constant $c$ such that*

$$\bar{y} = \chi_2(s(x-c), t(x-c))$$

*is a rational general solution of $F(x, y, y') = 0$.*

# Proof of the theorem

Assume that $(s(x), t(x))$ is a rational general solution of the associated system (5).

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$$R = \operatorname{prem}(G, F)$$

be the differential pseudo remainder of $G$ with respect to $F$. We have to prove that $R = 0$.

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$$R = \text{prem}(G, F)$$

be the differential pseudo remainder of $G$ with respect to $F$. We have to prove that $R = 0$. Note that the order of $R$ is 1 and

$$R(x, \bar{y}, \bar{y}') = 0$$

$$\text{where } (x, \bar{y}, \bar{y}') = \mathcal{P}(s(x-c), t(x-c)).$$

Assume that $R \neq 0$. Consider

$$R(\mathcal{P}(s,t)) = R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = \frac{W(s,t)}{Z(s,t)} \in \overline{\mathbb{Q}}(s,t).$$

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$$R(x, \bar{y}, \bar{y}') = 0 \Longrightarrow W(s(x-c), t(x-c)) = 0.$$

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$$R(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) = 0.$$

Since $F$ is irreducible and $\deg_{y'} R < \deg_{y'} F$, we have $R = 0$. Therefore, $\bar{y}$ is a rational general solution of $F(x, y, y') = 0$.

# Algorithm

- Input: $F(x, y, y') = 0$,
  A proper parametrization $(\chi_1(s,t), \chi_2(s,t), \chi_3(s,t)) \in \overline{\mathbb{Q}}(s,t)$ of $F(x, y, z) = 0$
- Output: A rational general solution of $F(x, y, y') = 0$.
  1. Compute $f_1(s,t), f_2(s,t), g(s,t)$ as in (4)
  2. Solve the associated system of ODEs for a rational general solution $(s(x), t(x))$
  $$\begin{cases} s' = \dfrac{f_1(s,t)}{g(s,t)} \\ t' = -\dfrac{f_2(s,t)}{g(s,t)} \end{cases}$$
  3. Compute the constant $c := \chi_1(s(x), t(x)) - x$
  4. Return $y = \chi_2(s(x-c), t(x-c))$.

# Example

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The associated system is

$$\begin{cases} s' = \dfrac{1}{2} \\ t' = 1. \end{cases}$$

Solving this associated system we obtain a rational general solution

$$s(x) = \frac{x}{2} + c_2, t(x) = x + c_1$$

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for arbitrary constants $c_1, c_2$. Therefore,

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and the rational general solution of $F(x, y, y') = 0$ is

$$\bar{y} = -4s^2(x - c_1)[2s(x - c_1) - t(x - c_1)] = -C(x + C)^2$$

where $C = 2c_2 - c_1$ is an arbitrary constant.

Thank you for your attention!