Regularization of linear integral equations with noisy data and noisy operator

Ismael Rodrigo Bleyer

Prof. Dr. Ronny Ramlau

Johannes Kepler Universität - Linz

Braunschweig - Sep 6-7, 2010.

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

JKU
JOHANNES KEPLER
UNIVERSITY LINZ

supported by
FWF
Der Wissenschaftsfonds.

Overview

- Introduction
- Proposed method
- Main results
- Computational aspects
- Numerical illustration
- Ongoing and future work

Overview

Basic problem

Consider a linear equation (discrete)

$$A_0 x = y_0$$

where $A_0 \in \mathbb{R}^{m \times n}$, $y_0 \in \mathscr{R}(A_0) \subset \mathbb{R}^m$ and seek a solution $x \in \mathbb{R}^n$.

- **noisy data** $y_\delta \in \mathbb{R}^m$ instead of $y_0 \in \mathscr{R}(A_0)$,

$$\|y_0 - y_\delta\| \leq \delta.$$

- **noisy operator** $A_\epsilon \in \mathbb{R}^{m \times n}$ instead of $A_0 \in \mathbb{R}^{m \times n}$,

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Main ingredients

1. minimize error: LS or TLS ?
   - Least squares

$$\begin{array}{ll} \text{minimize} & \|y - y_\delta\|_2 \\ \text{subject to} & y \in \mathscr{R}(A) \end{array}$$

   - Total least squares

$$\begin{array}{ll} \text{minimize} & \|[A, y] - [A_\epsilon, y_\delta]\|_F \\ \text{subject to} & y \in \mathscr{R}(A) \end{array}$$

$$\boxed{Ax = y}$$

2. regularization technique (due ill-posedness):
   - solution x apply $\|x\|$, $\|Dx\|$ or $\mathcal{R}(x)$, $\mathcal{R}$ convex functional.
   - solution (A, x) suitable $\|(A, x)\| := \|A\| + \|x\|$ - with two regularization parameters: **double regularization**.

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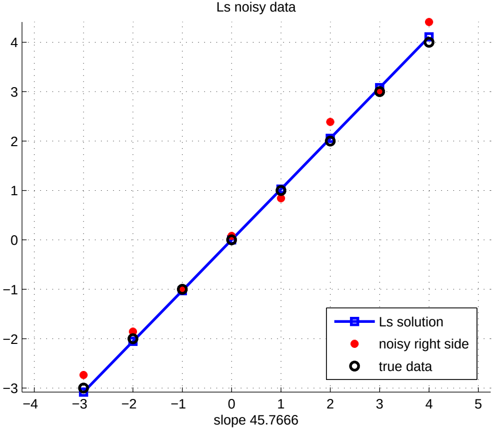
Illustration

Solve 1D problem: $xm = y$, find the slope m.

Cases:

1. y_δ
2. x_ϵ
3. y_δ, x_ϵ



Example: $m = 1$ or $\arctan(1) = 45^o$.

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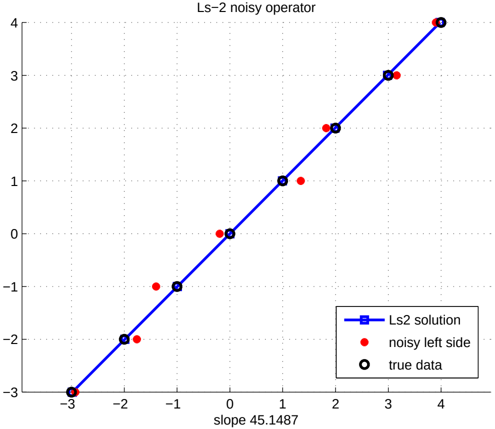

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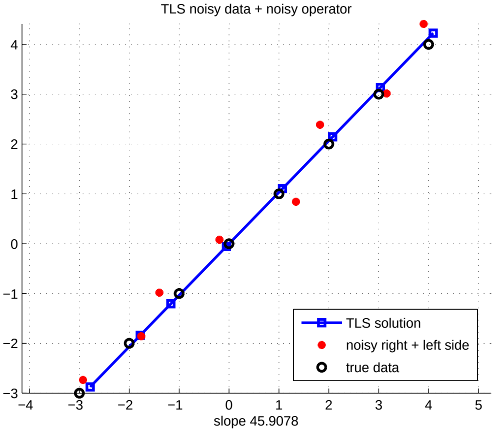

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Overview

In our context

Consider a (linear) ill-posed problems of the form

$$K_0 f = g_0 \, ,$$

where K_0 is a **integral operator**

$$\begin{aligned} K_0 : \quad \mathcal{U}_2 \quad &\longrightarrow \quad \mathcal{H} \\ f \quad &\longmapsto \quad \boxed{g_0 = K_0 f} \, , \end{aligned}$$

where

$$\big(K_0 f\big)(s) := \int_\Omega k_0(s,t) f(t) dt \, .$$

The operator K_0 is generated by kernel function $k_0 \in \mathcal{U}_1$.

Difficulties

Measurements:

- instead of $g_0 \in \mathscr{R}(K_0)$ we have **noisy data** $g_\delta \in Y$ with

$$\|g_0 - g_\delta\| \leq \delta.$$

- instead of $k_0 \in \mathcal{U}_1$ we have a **noisy operator** $k_\epsilon \in \mathcal{U}_1$ where

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Inverse problem:

- given g_δ and k_ϵ find a pair (k, f).

$$Kf = K(k, f) \qquad \text{Nonlinear!}$$

Idea: combine dR + TLS (double regularization with TLS)

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"Some mathematicians still have a kind of fear whenever they encounter a Fredholm integral equation of the first kind".

Integral operator $+$ kernel
($k \in L_2(\Omega^2)$ or $k \in C(\Omega^2)$)
$\Downarrow$
compact and **ill-posed**

Francesco Tricomi (1897-1978)

Motivated by previous ideas, we proposed **dR-TLS** method

$$\text{minimize} \;\; T(k,f) := \frac{1}{2} J(k,f) + \frac{\alpha}{2} \left\| L f \right\|^2_{L_2(\Omega)} + \beta \mathcal{R}(k) \, , \qquad (1)$$

where

$$J(k,f) = \left\| K(k,f) - g_\delta \right\|^2_{L_2(\Omega)} + \tau \left\| k - k_\epsilon \right\|^2_{L_2(\Omega^2)} \, ,$$

α, β are the regularization parameters, τ is a weight parameter and $\mathcal{R} : X \to [0, +\infty]$ is

- proper **convex** function and
- weak lower semi-continuous (**w-lsc**).

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Main results: theoretical

Proposition

Let T be the functional defined on (1) and L be a **positive defined** *operator. Then T is* **positive**, **weak lower semi-continuous** *and* **coercive** *functional.*

Theorem (existence)

Let the assumptions of Proposition 1 hold. Then there exists a **global minimum** *of*

$$\text{minimize } T(k, f) \, .$$

Theorem (stability)

- $\delta_j \to \delta$ *and* $\epsilon_j \to \epsilon$
- $g_{\delta_j} \to g_\delta$ *and* $k_{\epsilon_j} \to k_\epsilon$
- $\alpha, \beta > 0$
- (k^j, f^j) *is a minimizer of* T *with* g_{δ_j} *and* k_{ϵ_j}
- *Then there* **exists** *a convergent subsequence of* $(k^j, f^j)_j$

$$(k^{j_m}, f^{j_m}) \longrightarrow (\bar{k}, \bar{f})$$

where $(\bar{k}, \bar{f})$ *is a minimizer of* T *with* $g_\delta, k_\epsilon, \alpha$ *and* β.

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For convergence results we need to define

Definition

We call $(k^\dagger, f^\dagger)$ a Φ-**minimizing solution** if

$$(k^\dagger, f^\dagger) = \underset{(k,f)}{\arg\min} \left\{ \Phi(k,f) \;\middle|\; K(k,f) = g_0, k = k_0 \right\}.$$

where the equalities above hold almost everywhere.

We define the convex functional

$$\Phi(k,f) := \frac{1}{2}\left\| L\ f \right\|^2 + \eta \mathcal{R}(k)$$

where the parameter η represents the different scaling of f and k.

Theorem (convergence)

- $\delta_j \to 0$ *and* $\epsilon_j \to 0$
- $\left\|g_{\delta_j} - g_0\right\| \leq \delta_j$ *and* $\left\|k_{\epsilon_j} - k_0\right\| \leq \epsilon_j$
- $\alpha_j = \alpha(\epsilon_j, \delta_j)$ *and* $\beta_j = \beta(\epsilon_j, \delta_j)$, *s.t.* $\alpha_j \to 0$, $\beta_j \to 0$,

$$\lim_{j\to\infty} \frac{\delta_j^2 + \tau\epsilon_j^2}{\alpha_j} = 0 \quad \textit{and} \quad \lim_{j\to\infty} \frac{\beta_j}{\alpha_j} = \eta$$

- (k^j, f^j) *is a minimizer of* T *with* g_{δ_j}, k_{ϵ_j}, α_j *and* β_j
- *Then there* **exists** *a convergent subsequence of* $(k^j, f^j)_j$

$$(k^{j_m}, f^{j_m}) \longrightarrow (k^\dagger, f^\dagger)$$

where $(k^\dagger, f^\dagger)$ *is a* Φ-**minimizing solution**.

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$$(k^{j_m}, f^{j_m}) \longrightarrow (k^\dagger, f^\dagger)$$

where $(k^\dagger, f^\dagger)$ *is a* Φ-**minimizing solution**.

Remark

As a direct consequence, the choice $\beta = \eta\alpha$ satisfies the previous theorem. So, defining

$$\Phi(k,f) = \frac{1}{2}\left\|L\,f\right\|^2 + \eta\mathcal{R}\left(k\right)$$

and

$$\begin{array}{rcccc} B(k,f) & : & L_2(\Omega^2)\times L_2(\Omega) & \longrightarrow & L_2(\Omega)\times L_2(\Omega) \\ & & (k,f) & \longmapsto & (K(k,f),k) \end{array}$$

we can rewrite (1) as

$$\text{minimize}\,\frac{1}{2}\left\|B(k,f) - (g_\delta, k_\epsilon)\right\|^2 + \alpha\Phi(k,f)$$

where $\left\|(x_1,y_1)-(x_2,y_2)\right\|_\tau^2 := \left\|x_1 - x_2\right\|^2 + \tau\left\|y_1 - y_2\right\|^2$.

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Optimality condition

If the pair $(\bar{k}, \bar{f})$ is a minimizer of $T(k, f)$, then $0 \in \partial T(\bar{k}, \bar{f})$.

$$\partial\Big(\underbrace{J(\bar{k}, \bar{f}) + \alpha \left\| L\bar{f} \right\|^2}_{\text{Fréchet differentiable}} + \beta\,\mathcal{R}(\bar{k})\Big) = h'(k, f)(u, v) + \beta\,\partial\mathcal{R}(\bar{k})$$

where

$$h'(k, f)(u, v) = \left\langle \begin{bmatrix} (K_f^* K_f + \tau I)k - (\tau k_\epsilon + K_f^* g_\delta) \\ (K_k^* K_k + \alpha L^* L)f - K_k^* g_\delta \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{L_2 \times L_2}$$

for all $(u, v) \in L_2(\Omega^2) \times L_2(\Omega)$ and

$$\partial\mathcal{R}(\bar{k}) = \{\xi \in L_2(\Omega^2) \mid \mathcal{R}(k) \geq \mathcal{R}(\bar{k}) + \langle \xi, k - \bar{k} \rangle\}$$

for all $k \in L_2(\Omega^2) \cap \mathscr{D}(R)$.

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for all $k \in L_2(\Omega^2) \cap \mathscr{D}(R)$.

Which convex function?

For the choice: $\mathcal{R}(k) = \left\|k\right\|_{L_1(\Omega^2)}$ we know

$$\partial\mathcal{R}(k(s,t)) = \operatorname{sgn}(k(s,t)) \text{ for a.e. } (s,t) \in \Omega^2$$

where

$$\operatorname{sgn}(z) = \begin{cases} \left\{\frac{z}{|z|}\right\} & \text{if } z \neq 0 \\ \{\xi \in \mathbb{C} \ \mid \ |\xi| \leq 1\} & \text{otherwise} \end{cases}$$

Necessary optimality condition: if $(\bar{k}, \bar{f})$ minimizes (1) then

$$\begin{cases} (K_{\bar{k}}^* K_{\bar{k}} + \alpha L^* L)\bar{f} = K_{\bar{k}}^* g_\delta \\ (K_{\bar{f}}^* K_{\bar{f}} + \tau I)\bar{k} = K_{\bar{f}}^* g_\delta + \tau k_\epsilon - \beta \operatorname{sgn}(\bar{k}) \end{cases}$$

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Remark:

- first equation: f depends of k and α.

$$f = \left(K_k^* K_k + \alpha L^* L\right)^{-1} K_k^* g_\delta.$$

- second equation: soft-shrinkage operator

$$\begin{aligned} k &= k_\epsilon - \frac{1}{\tau} K_f^* \left(g_\delta - K_f k\right) - \frac{\beta}{\tau} \operatorname{sgn}(k) \\ &= \mathcal{S}_{\frac{\beta}{\tau}} \left(k_\epsilon - \frac{1}{\tau} K_f^* \left(g_\delta - K_f k\right) \right) \end{aligned}$$

- partial derivatives of functional T

$$\frac{\partial T(k, f|k)}{\partial f} \quad \text{and} \quad \frac{\partial T(k, f|f)}{\partial k}$$

Idea: AL (Alternating minimization)

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Iterative shrinkage-thresholding algorithm

Require: $L, g_\delta, k_\epsilon, \tau$ and $k^0 \in L_2(\Omega^2) \cap L_1(\Omega^2)$

$n = 0$

repeat

  choose α and β

  $k^{n+1} = \mathcal{S}_{\beta/\tau}\left(k_\epsilon - \frac{1}{\tau} K_{f^n}^*(g_\delta - K_{f^n} k^n)\right)$

until convergence

Alternating minimization algorithm

Require: $L, g_\delta, k_\epsilon, \tau$ and $(k^0, f^0) \in L_2(\Omega^2) \cap L_1(\Omega^2) \times L_2(\Omega)$

$n = 0$

repeat

  $\text{descend}_{f^n} T(k^n, f^n | k^n)$

  $\text{descend}_{k^n} T(k^n, f^n | f^{n+1})$

until convergence

Iterative shrinkage-thresholding algorithm

Require: $L, g_\delta, k_\epsilon, \tau$ and $k^0 \in L_2(\Omega^2) \cap L_1(\Omega^2)$

1: $n = 0$

2: **repeat**

3: choose α and β

4: $k^{n+1} = \mathcal{S}_{\beta/\tau}\left(k_\epsilon - \frac{1}{\tau}K^*_{f^n}(g_\delta - K_{f^n}k^n)\right)$

5: **until** convergence

Alternating minimization algorithm

Require: $L, g_\delta, k_\epsilon, \tau$ and $(k^0, f^0) \in L_2(\Omega^2) \cap L_1(\Omega^2) \times L_2(\Omega)$

1: $n = 0$

2: **repeat**

3: $\mathrm{descend}_{f^n}\, T(k^n, f^n | k^n)$

4: $\mathrm{descend}_{k^n}\, T(k^n, f^n | f^{n+1})$

5: **until** convergence

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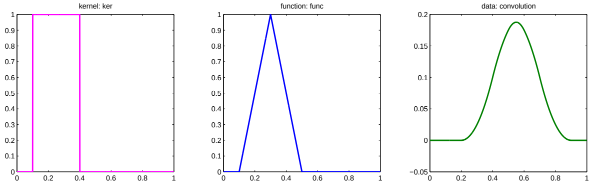
First numerical result

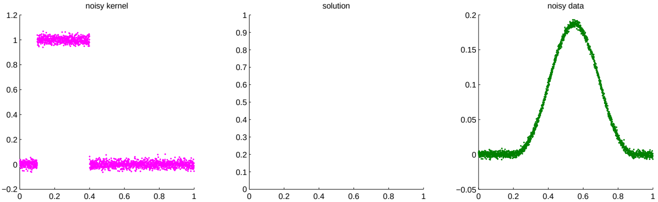
Convolution in 1D

$$\int_{\Omega} k(s-t)f(t)dt = g(s)$$



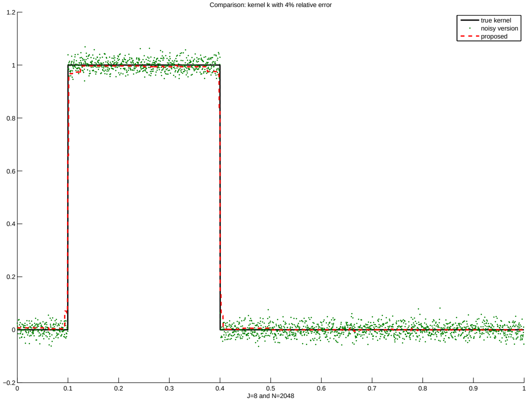
- space: $\Omega = [0,1]$, discretization: $N = 2048$ points
- initial guess: $k^0 = k_\epsilon$, $\tau = 1.0$
- relative error: 4% and 3%

Noisy kernel and data

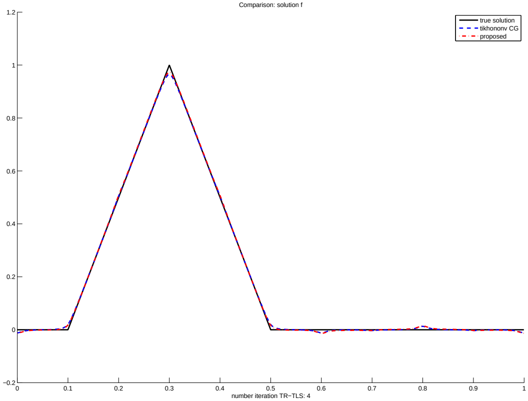


Tolerance: 10^{-7}. Iterations: Proposed 3 and Tikhonov (CG) 21.

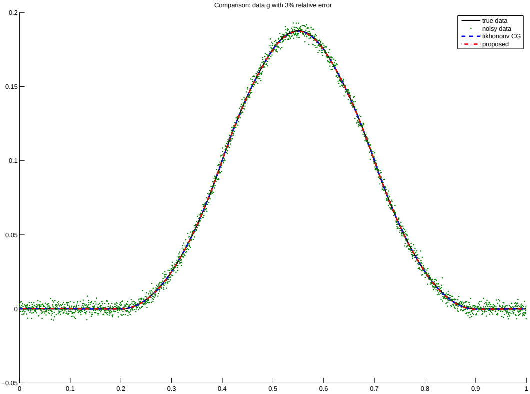
$\left\|k^n - k^{n+1}\right\|_2$	$\left\|f^n - f^{n+1}\right\|_2$	$\frac{\left\|k^n - k_0\right\|_2}{\left\|k_0\right\|_2}$	$\frac{\left\|f^n - f_0\right\|_2}{\left\|f_0\right\|_2}$
1.4479	16.4736	0.0194	0.0141
0.0001	0.1175	0.0194	0.0138
0.0000	0.0000	0.0194	0.0138



noisy kernel	0.0586
proposed	0.0194



Tikhonov	1.9954e-04
proposed	1.9013e-04



Tikhonov	5.6513e-06
proposed	5.7668e-06

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Ongoing and future work

- design an algorithm
- study of source conditions;
- prove convergence rates;
- how to choose the best regularization parameter?
- *a priori* and *a posteriori* choice;
- implementations and numerical experiments (2D)

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Thank you for your kind attention!

Questions?