Regularization of linear integral equations with noisy data and noisy operator

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

Ismael Rodrigo Bleyer, DK, Linz
Ronny Ramlau, JKU, Linz

This project is funded by DK Computational Mathematics

JOHANNES KEPLER UNIVERSITY LINZ

Introduction

Regularization methods for solving a linear ill-posed problems of the form

$$K_0 f = g_0 \,, \tag{1}$$

where $K_0 : \mathcal{U} \longrightarrow \mathcal{H}$ is a bounded linear operator between infinite dimensional real Hilbert spaces $\mathcal{U}$ and $\mathcal{H}$, have been extensively investigated when there exists noisy measurements on the data:

- instead of $g_0 \in \mathscr{R}(K_0)$ we have a noisy right hand side $g_\delta \in \mathcal{H}$ with

$$\|g_0 - g_\delta\| \leq \delta. \tag{2}$$

In real applications both elements K_0 and g_0 in (1) are contaminated by some noise. For a more realistic situation we also assume:

- instead of $K_0 \in \mathscr{L}(\mathcal{U},\mathcal{H})$ we have a noisy operator $K_\epsilon \in \mathscr{L}(\mathcal{U},\mathcal{H})$ with

$$\|K_0 - K_\epsilon\| \leq \epsilon. \tag{3}$$

The numerical treatment of ill-posed problem (1) with noisy data (2) and (3) requires the application of special regularization techniques.

Tikhonov regularization is the most widely applied methods for solving ill-posed problems [6, 1].

$$\begin{array}{ll} \text{minimize} & \|K_\epsilon f - g_\delta\|^2 \\ \text{subject to} & \|Lf\|^2 \leq M, \end{array}$$

Regularized total least square is a method based on TLS [3, 5], adding a stabilization term with respect to the solution f.

$$\begin{array}{ll} \text{minimize} & \|K - K_\epsilon\|^2 + \|g - g_\delta\|^2 \\ \text{subject to} & \begin{cases} Kf = g \\ \|Lf\|^2 \leq M. \end{cases} \end{array}$$

Proposed method

Our approach is based on R-TLS method [2, 4]. We propose a solution of (1) for the case: $K_0 : L^2(\Omega) \to L^2(\Omega)$ is an integral equation

$$\Big(K_0 f\Big)(s) = \int_\Omega k_0(s,t) f(t) dt$$

generated by the kernel $k_0(\cdot,\cdot) \in L^2(\Omega^2)$ and $f(\cdot) \in L^2(\Omega)$, with $\Omega \subseteq \mathbb{R}^d$.

The approximated solution is the pair (k,f) which solves the following minimization problem

$$\text{minimize} \;\; T(k,f) := \frac{1}{2} J(k,f) + \beta\,\mathcal{R}(k) \;, \tag{4a}$$

where

$$J(k,f) = \|K(k,f) - g_\delta\|^2_{L^2(\Omega)} + \alpha\|Lf\|^2_{L^2(\Omega)} + \tau\|k - k_\epsilon\|^2_{L^2(\Omega^2)} \,, \tag{4b}$$

α, β are the regularization parameters to be chosen properly, τ is a weight parameter and

$$\mathcal{R}(k) = \|k\|_{L^1(\Omega^2)} \,. \tag{4c}$$

Main results

We present some results about the quality of the method introduced in (4), for instance: existence, stability and convergence.

Proposition 1. *Let T be the functional defined on (4) and L be a positive defined operator. Then T is positive, weak lower semicontinuous and coercive functional.*

Theorem: Existence

Let the assumptions of Proposition 1 hold. Then there exists a global minimum of

$$\text{minimize} \;\; T(k,f) \,.$$

Theorem: Stability

Let $\alpha, \beta > 0$ the regularization parameters, L be a positive defined operator and $(g_{\delta_j})_j$, $(k_{\epsilon_j})_j$ sequences where $g_{\delta_j} \longrightarrow g_\delta$ and $k_{\epsilon_j} \longrightarrow k_\epsilon$. Associate with the noisy data and noisy kernel compute a sequence of solutions $(k^j, f^j)_j$, where (k^j, f^j) is a minimizer of T with g_{δ_j} and k_{ϵ_j} replaced by g_δ and k_ϵ respectively. Then there exists a convergent subsequence of $(k^j, f^j)_j$ and the limit of every convergent subsequence is a minimizer of functional T.

Definition. *We call $(k^\dagger, f^\dagger)$ a $\frac{1}{2}\|L\ \cdot\|^2 + \eta\|\cdot\|_1$- minimizing solution if*

$$(k^\dagger, f^\dagger) = \underset{(k,f)}{\arg\min} \left\{ \frac{1}{2}\|Lf\|^2 + \eta\|k\|_1 \;\middle|\; K(k,f) = g_0, k = k_0 \right\}.$$

Theorem: Convergence

Let the noisy data g_{δ_j} and noisy kernel k_{ϵ_j} with $\|g_{\delta_j} - g_0\| \leq \delta_j$ and $\|k_{\epsilon_j} - k_0\| \leq \epsilon_j$. Let the regularization parameters $\alpha_j = \alpha(\epsilon_j, \delta_j)$ and $\beta_j = \beta(\epsilon_j, \delta_j)$ satisfy $\alpha_j \to 0$, $\beta_j \to 0$,

$$\lim_{j\to\infty} \frac{\delta_j^2 + \tau\epsilon_j^2}{\alpha_j} = 0 \quad \text{and} \quad \lim_{j\to\infty} \frac{\beta_j}{\alpha_j} = \eta$$

for some $0 < \eta < \infty$, as long as the sequence of noise level $\epsilon_j \to 0$, $\delta_j \to 0$.
Let the sequence $(k^j, f^j)_j := \Big(k^{\delta_j,\epsilon_j}_{\alpha_j,\beta_j}, f^{\delta_j,\epsilon_j}_{\alpha_j,\beta_j}\Big)_j$ be the solution of the (4) with respective noisy data g_{δ_j}, noisy kernel k_{ϵ_j}, regularization parameters α_j, β_j and weight parameter τ. Then there exists a convergent subsequence of $(k^j, f^j)_j$. The limit of every convergent subsequence is a $\frac{1}{2}\|L\ \cdot\|^2 + \eta\|\cdot\|_1$- minimizing solution. Moreover, if the minimizer $(k^\dagger, f^\dagger)$ is unique, then

$$\lim_{j\to\infty} (k^j, f^j) = (k^\dagger, f^\dagger) \,.$$

Algorithm

Computing the first order optimality condition for the functional T, one can show the following result.

Theorem. *The solution (k,f) of (4) satisfies the system equation*

$$\begin{cases} \Big(K_k^* K_k + \alpha L^* L\Big)(f)(t) = K_k^*(g_\delta)(t) \\ \Big(K_f^* K_f + \tau I\Big)(k)(s,t) = \Big(K_f^*(g_\delta) + \tau k_\epsilon\Big)(s,t) - \beta\,\mathrm{sgn}(k(s,t)), \end{cases}$$

for a.e. $(s,t) \in \Omega \times \Omega$.

Aiming to solve such system of equations iteratively, we propose the iterative shrinkage-thresholding algorithm:

Require: $L, g_\delta, k_\epsilon, \tau$ and $k^0 \in L^2(\Omega^2) \cap L^1(\Omega^2)$

1: $n = 0$
2: **repeat**
3: choose α and β
4: $k^{n+1} = \mathcal{S}_\beta\Big(k^n + K^*_{f^\alpha_\delta(k^n)}(g_\delta - K_{f^\alpha_\delta(k^n)}k^n) + \tau(k_\epsilon - k^n)\Big)$
5: **until** convergence

where the soft-shrinkage operator $\mathcal{S}_\beta(\cdot)$ is defined as

$$\mathcal{S}_\beta(x) = \begin{cases} x - \beta\dfrac{x}{|x|} & , |x| > \beta \\ 0 & , |x| \leq \beta \end{cases}$$

and for each iteration we solve the linear system

$$f^\alpha_\delta(k^n) = \Big(K^*_{k^n} K_{k^n} + \alpha L^* L\Big)^{-1} K^*_{k^n} g_\delta$$

for some regularization parameter α.

References

[1] Heinz W. Engl, Martin Hanke, and Andreas Neubauer. *Regularization of Inverse Problems*. Kluwer Academic Publishers, Dordrecht, 2000.
[2] Gene H. Golub, Per Christian Hansen, and Dianne P. O'leary. Tikhonov regularization and total least squares. *SIAM J. Matrix Anal. Appl*, 21:185–194, 1999.
[3] Gene H. Golub and Charles F. Van Loan. An analysis of the total least squares problem. *SIAM J. Numer. Anal.*, 17(6):883–893, 1980.
[4] Shuai Lu, Sergei V. Pereverzev, and Ulrich Tautenhahn. Regularized total least squares: computational aspects and error bounds. Technical Report 30, Ricam, Linz, Austria, 2007.
[5] I. Markovsky and S. Van Huffel. Overview of total least squares methods. *Signal Processing*, 87:2283–2302, 2007.
[6] Andrei Nikolaevich Tikhonov and V. A. Arsenin. *Solution of Ill-posed Problems*. Winston & Sons, Washington, 1977.