Regularization of linear integral equations with noisy data and noisy operator

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Overview

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General problem

Consider a linear ill-posed problems of the form

$$K_0 f = g_0 \,,$$

where $K_0 : \mathcal{U} \to \mathcal{H}$ is a bounded linear operator between infinite dimensional real Hilbert spaces $\mathcal{U}$ and $\mathcal{H}$.

- instead of $g_0 \in \mathscr{R}(K_0)$ we have **noisy data** $g_\delta \in Y$ with

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- instead of $K_0 \in \mathscr{L}(\mathcal{U}, \mathcal{H})$ we have a **noisy operator** $K_\epsilon \in \mathscr{L}(\mathcal{U}, \mathcal{H})$ where

$$\|K_0 - K_\epsilon\| \leq \epsilon.$$

Difficulties

Consider (linear) integral operator K_0, function spaces $\mathcal{U}$ and $\mathcal{H}$

$$\begin{array}{rccc} K_0 : & \mathcal{U} & \longrightarrow & \mathcal{H} \\ & f & \longmapsto & \boxed{g_0 = K_0 f}, \end{array}$$

where

$$\big(K_0 f\big)(s) := \int_\Omega k_0(s,t) f(t) dt .$$

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Inverse problem: given g_0 find function f.

Integral operator + kernel ($k \in L^2(\Omega^2)$, continuous)

$\Downarrow$

compact and ill-posed

Measurements: data g_δ and kernel k_ϵ.

Overview

How to solve?

- **Tikhonov regularization** is the most widely applied methods for solving ill-posed problems

$$\begin{aligned} \text{minimize} \quad & \left\| K_\epsilon f - g_\delta \right\|^2 \\ \text{subject to} \quad & \left\| Lf \right\|^2 \leq M, \end{aligned}$$

- **Regularized total least square** is a method based on TLS Golub and Van Loan [1980], adding a stabilization term with respect to the solution f.

$$\begin{aligned} \text{minimize} \quad & \left\| K - K_\epsilon \right\|^2 + \left\| g - g_\delta \right\|^2 \\ \text{subject to} \quad & \begin{cases} Kf = g \\ \left\| Lf \right\|^2 \leq M. \end{cases} \end{aligned}$$

Remark: discretized, finite dimension problem.

Main idea

Based on R-TLS

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$$\begin{array}{ll} \text{minimize} & \left\|K - K_\epsilon\right\|^2 + \left\|Kf - g_\delta\right\|^2 \\ \text{subject to} & \begin{cases} \left\|K\right\| \le N \\ \left\|Lf\right\|^2 \le M\,. \end{cases} \end{array}$$

This problem can be rewritten as an unrestricted minimization problem

$$\text{minimize} \left\|Kf - g_\delta\right\|^2 + \alpha\left\|Lf\right\|^2 + \left\|K - K_\epsilon\right\|^2 + \beta\left\|K\right\|,$$

where α and β are called regularization parameters.

Remark: $K := K(k, f)$ is a bilinear operator and

$$\left\|K(k,f)\right\|^2_{L^2(\Omega)} \le \left\|k\right\|^2_{L^2(\Omega^2)}\left\|f\right\|^2_{L^2(\Omega)} \quad , \quad \left\|K\right\|^2_{L^2(\Omega)\to L^2(\Omega)} \le \left\|k\right\|^2_{L^2(\Omega^2)}$$

Proposed method

In summary, we compute the approximate solution via minimization problem

$$\text{minimize} \;\; T(k,f) := \frac{1}{2} J(k,f) + \beta \mathfrak{R}(k) \, , \qquad (1)$$

where

$$J(k,f) = \left\| K(k,f) - g_\delta \right\|^2_{L^2(\Omega)} + \alpha \left\| Lf \right\|^2_{L^2(\Omega)} + \tau \left\| k - k_\epsilon \right\|^2_{L^2(\Omega^2)} ,$$

α, β are the regularization parameters, τ is a weight parameter and

$$\mathfrak{R}(k) = \left\| k \right\|_{L^1(\Omega^2)} .$$

Overview

Main results: theoretical

Proposition

*Let T be the functional defined on (1) and L be a positive defined operator. Then T is **positive**, **weak lower semi-continuous** and **coercive** functional.*

Theorem (existence)

*Let the assumptions of Proposition 1 hold. Then there exists a **global minimum** of*

$$\text{minimize} \quad T(k, f) \ .$$

Theorem (stability)

*Let $\alpha, \beta > 0$ the regularization parameters, L be a **positive defined** operator and $(g_{\delta_j})_j$, $(k_{\epsilon_j})_j$ sequences where $g_{\delta_j} \to g_\delta$ and $k_{\epsilon_j} \to k_\epsilon$. Associate with the noisy data and noisy kernel compute a sequence of solutions $(k^j, f^j)_j$, where (k^j, f^j) is a minimizer of T with g_{δ_j} and k_{ϵ_j} replaced by g_δ and k_ϵ respectively. Then there **exists** a convergent subsequence of $(k^j, f^j)_j$ and the limit of every convergent subsequence is a **minimizer** of functional T.*

$$(k^{j_m}, f^{j_m}) \longrightarrow (\bar{k}, \bar{f}) := \arg\min T(k, f)$$

Theorem (stability)

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Definition

We call $(k^\dagger, f^\dagger)$ a $\frac{1}{2}\|L\,\cdot\|^2 + \eta\|\cdot\|_1$- **minimizing solution** if

$$(k^\dagger, f^\dagger) = \arg\min_{(k,f)} \left\{ \frac{1}{2}\|Lf\|^2 + \eta\|k\|_1 \;\middle|\; K(k,f) = g_0, k = k_0 \right\}.$$

Theorem (Convergence)

Let the noisy data g_{δ_j} *and noisy kernel* k_{ϵ_j} *with* $\|g_{\delta_j} - g_0\| \leq \delta_j$ *and* $\|k_{\epsilon_j} - k_0\| \leq \epsilon_j$. *Let the* ***regularization parameters*** $\alpha_j = \alpha(\epsilon_j, \delta_j)$ *and* $\beta_j = \beta(\epsilon_j, \delta_j)$ *satisfy* $\alpha_j \to 0$, $\beta_j \to 0$,

$$\lim_{j\to\infty} \frac{\delta_j^2 + \tau\epsilon_j^2}{\alpha_j} = 0 \quad \text{and} \quad \lim_{j\to\infty} \frac{\beta_j}{\alpha_j} = \eta$$

for some $0 < \eta < \infty$, *as long as the sequence of noise level* $\epsilon_j \to 0$, $\delta_j \to 0$.

Let the sequence $(k^j, f^j)_j := \left(k^{\delta_j,\epsilon_j}_{\alpha_j,\beta_j}, f^{\delta_j,\epsilon_j}_{\alpha_j,\beta_j}\right)_j$ *be the solution of the* (1) *with respective noisy data* g_{δ_j}, *noisy kernel* k_{ϵ_j}, *regularization parameters* α_j, β_j *and weight parameter* τ. *Then there exists a* ***convergent subsequence*** *of* $(k^j, f^j)_j$. *The limit of every convergent subsequence is a* $\frac{1}{2}\|L\cdot\|^2 + \eta\|\cdot\|_1$*-* ***minimizing solution***. *Moreover, if the minimizer* $(k^\dagger, f^\dagger)$ *is unique, then*

$$\lim (k^j, f^j) = (k^\dagger, f^\dagger).$$

Overview

Computational aspects

Optimality condition: if the pair $(\bar{k}, \bar{f})$ is a minimizer of $T(k, f)$, then

$$0 \in \partial T\left(\bar{k}, \bar{f}\right) = \partial\left(J\left(\bar{k}, \bar{f}\right) + \beta \mathfrak{R}\left(\bar{k}\right)\right)$$

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We know

$$J'(k,f)(u,v) = 2\left\langle \begin{bmatrix} (K_f^* K_f + \tau I)k - (\tau k_\epsilon + K_f^* g_\delta) \\ (K_k^* K_k + \alpha L^* L)f - K_k^* g_\delta \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle_{L^2(\Omega^2) \times L^2(\Omega)}$$

and [Justen and Ramlau, 2009]

$$\partial \mathcal{R}(k(s,t)) = \operatorname{sgn}(k(s,t)) \text{ for a.e. } (s,t) \in \Omega^2$$

where

$$\operatorname{sgn}(z) = \begin{cases} \left\{\frac{z}{|z|}\right\} & \text{if } z \neq 0 \\ \{\xi \in \mathbb{C} \mid |\xi| \leq 1\} & \text{otherwise} \end{cases}$$

Candidates for a minimizer of our problem have to fulfill the optimality condition:

$$\begin{cases} (K_k^* K_k + \alpha L^* L) f = K_k^* g_\delta \\ (K_f^* K_f + \tau I) k = K_f^* g_\delta + \tau k_\epsilon - \beta \operatorname{sgn}(k) \end{cases}$$

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Remark: iterative development

- first equation: f depends of k and α.

$$f_\delta^\alpha(k) = \left(K_k^* K_k + \alpha L^* L\right)^{-1} K_k^* g_\delta .$$

- second equation: add k in both sides

$$k^{n+1} = \left(k^n + K_f^* g_\delta + \tau k_\epsilon - (K_f^* K_f + \tau I) k^n\right) - \beta \,\mathrm{sgn}\left(k^n\right) .$$

Algorithm

Such formulation leads us to apply the soft-shrinkage operator $\mathcal{S}_\beta(\,\cdot\,)$, defined as

$$\mathcal{S}_\beta(x) = \begin{cases} x - \beta\dfrac{x}{|x|} & , \ |x| > \beta \\ 0 & , \ |x| \le \beta\,. \end{cases}$$

We update k as following way

$$k^{n+1} = \mathcal{S}_\beta\big(k^n + K_f^* g_\delta + \tau k_\epsilon - (K_f^* K_f + \tau I)k^n\big)\,.$$

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Require: $L, g_\delta, k_\epsilon, \tau$ and $k^0 \in L^2(\Omega^2) \cap L^1(\Omega^2)$

1: $n = 0$
2: **repeat**
3: choose α and β
4: $k^{n+1} = \mathcal{S}_\beta\big(k^n + K^*_{f^\alpha_\delta(k^n)}(g_\delta - K_{f^\alpha_\delta(k^n)}k^n) + \tau(k_\epsilon - k^n)\big)$
5: **until** convergence

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Thank you for your attention!