A Double Regularization Approach for Inverse Problems with Noisy Data and Inexact Operator

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Overview

- Introduction
- Proposed method: DBL-RTLS
- Computational aspects
- Numerical illustration

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Inverse problems

"*Inverse problems* are concerned with determining causes for a desired or an observed effect" [Engl, Hanke, and Neubauer, 2000]

Consider a linear operator equation

$$Ax = y.$$

Inverse problems most oft do not fulfill **Hadamard**'s postulate [1902] of well posedness (**existence, uniqueness** and **stability**).

Computational issues: observed effect has measurement *errors* or perturbations caused by *noise*.

1st Case: noisy data

Solve $Ax = y_0$ out of the measurement y_δ with $\|y_0 - y_\delta\| \leq \delta$.
Need apply some **regularization** technique

$$\underset{x}{\text{minimize}} \|Ax - y_\delta\|^2 + \alpha \|Lx\|^2.$$

Tikhonov regularization

- fidelity term (based on LS);
- regularization parameter α;
- stabilization term (quadratic).

[Tikhonov, 1963, Phillips, 1962]

1st Case: noisy data

Solve $Ax = y_0$ out of the measurement y_δ with $\|y_0 - y_\delta\| \le \delta$.
Need apply some **regularization** technique

$$\underset{x}{\text{minimize}} \|Ax - y_\delta\|^2 + \alpha \mathcal{R}(x).$$

Tikhonov-type regularization

- fidelity term (based on LS);
- regularization parameter α;
- $\mathcal{R}$ is a *proper*, *convex* and *weakly lower semicontinuous functional*.

[Burger and Osher, 2004, Resmerita, 2005]

Subgradient

The *Fenchel subdifferential* of a functional $\mathcal{R} : \mathcal{U} \to [0, +\infty]$ at $\bar{u} \in \mathcal{U}$ is the set

$$\partial^F \mathcal{R}\,(\bar{u}) = \{\xi \in \mathcal{U}^* \mid \mathcal{R}(v) - \mathcal{R}(\bar{u}) \geq \langle \xi\, ,\, v - \bar{u} \rangle \; \forall v \in \mathcal{U}\}.$$

First in 1960 by Moreau & Rockafellar and extended by Clark 1973.

Optimality condition:

If $\bar{u}$ minimizes $\mathcal{R}$ then

$$0 \in \partial^F \mathcal{R}\,(\bar{u})$$

Example

Consider the function $\mathcal{R}(u) = |u|$

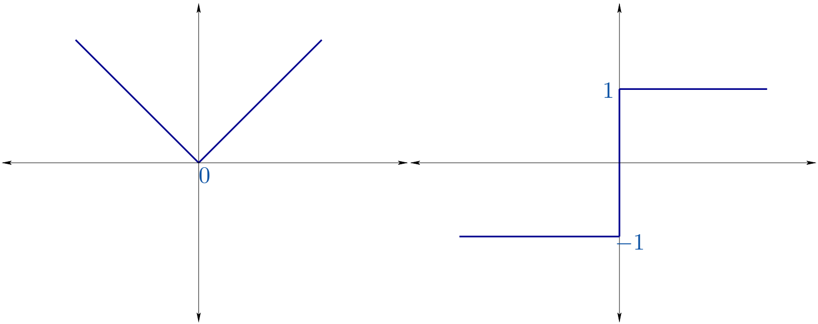

Figure: Function (left) and its subdifferential (right).

2nd Case: inexact operator and noisy data

Solve $A_0 x = y_0$ under the assumptions

(i) noisy data $\|y_0 - y_\delta\| \leq \delta$.

(ii) inexact operator $\|A_0 - A_\epsilon\| \leq \epsilon$.

What have been done so far?

- *Linear case* - based on TLS [Golub and Van Loan, 1980]:
  - **R-TLS**: Regularized TLS [Golub et al., 1999];
  - **D-RTLS**: Dual R-TLS [Lu et al., 2007].
- *Nonlinear case*: no publication (?)

LS: y_δ and A_0

$$\begin{array}{ll} \text{minimize}_y & \|y - y_\delta\|_2 \\ \text{subject to} & y \in \mathscr{R}(A_0) \end{array}$$

TLS: y_δ and A_ϵ

$$\begin{array}{ll} \text{minimize} & \|[A, y] - [A_\epsilon, y_\delta]\|_F \\ \text{subject to} & y \in \mathscr{R}(A) \end{array}$$

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Illustration

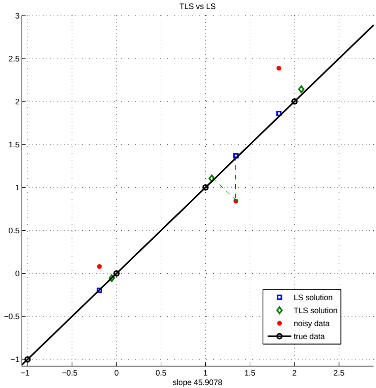
Solve 1D problem: $am = b$, find the slope m.

Given:

1. b_δ, a_ϵ (red)

Solution:

1. LS solution (blue)
2. TLS solution (green)



Example: $\arctan(1) = 45^o$ [Van Huffel and Vandewalle, 1991]

R-TLS

The R-TLS method [Golub, Hansen, and O'leary, 1999]

$$\text{minimize} \quad \left\|A - A_\epsilon\right\|^2 + \left\|y - y_\delta\right\|^2$$
$$\text{subject to} \quad \begin{cases} Ax = y \\ \left\|Lx\right\|^2 \leq M \,. \end{cases}$$

If the inequality constraint is active, then

$$(A_\epsilon^T A_\epsilon + \alpha L^T L + \beta I)\hat{x} = A_\epsilon^T y_\delta \text{ and } \left\|L\hat{x}\right\| = M$$

with $\alpha = \mu(1 + \left\|\hat{x}\right\|^2)$, $\beta = -\dfrac{\left\|A_\epsilon \hat{x} - y_\delta\right\|^2}{1 + \left\|\hat{x}\right\|^2}$ and $\mu > 0$ is the Lagrange multiplier.

Difficulty: requires a reliable bound M for the norm $\left\|Lx^\dagger\right\|^2$.

Overview

Consider the operator equation

$$B(k,f) = g_0$$

where B is a bilinear operator (nonlinear)

$$\begin{aligned} B: \quad \mathcal{U} \times \mathcal{V} &\longrightarrow \mathcal{H} \\ (k,f) &\longmapsto B(k,f) \end{aligned}$$

and B is characterized by a function k_0.

- $K\cdot = B(\tilde{k}, \cdot)$ compact linear operator for a fixed $\tilde{k} \in \mathcal{U}$
- $F\cdot = B(\cdot, \tilde{f})$ linear operator for a fixed $\tilde{f} \in \mathcal{V}$
- $\|B(k_0, \cdot)\|_{\mathcal{V}\to\mathcal{H}} \leq C\|k_0\|_{\mathcal{U}}$;
- $\|B(k,f)\|_{\mathcal{H}} \leq C\|k\|_{\mathcal{U}}\|f\|_{\mathcal{V}}$;

Example:

$$B(k,f)(s) := \int_\Omega k(s,t)f(t)dt\,.$$

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- $\left\| B(k, f) \right\|_{\mathcal{H}} \leq C \left\| k \right\|_{\mathcal{U}} \left\| f \right\|_{\mathcal{V}}$;

Example:

$$B(k, f)(s) := \int_{\Omega} k(s, t) f(t) dt \,.$$

We want to solve

$$B(k_0, f) = g_0$$

out of the measurements k_ϵ and g_δ with

(i) noisy data $\left\|g_0 - g_\delta\right\|_{\mathcal{H}} \leq \delta$.

(ii) inexact operator $\left\|k_0 - k_\epsilon\right\|_{\mathcal{U}} \leq \epsilon$.

We introduce the DBL-RTLS

$$\underset{k,f}{\text{minimize}} \;\; J(k, f) := T(k, f, k_\epsilon, g_\delta) + R(k, f)$$

where

- T measures of accuracy (closeness/discrepancy)
- R promotes stability.

DBL-RTLS

$$\underset{k,f}{\text{minimize}} \;\; J(k,f) := T(k,f,k_\epsilon,g_\delta) + R(k,f) \tag{1}$$

where

$$T(k,f,k_\epsilon,g_\delta) = \frac{1}{2}\left\|B(k,f) - g_\delta\right\|_{\mathcal{H}}^2 + \frac{\gamma}{2}\left\|k - k_\epsilon\right\|_{\mathcal{U}}^2$$

$$R(k,f) = \frac{\alpha}{2}\left\|Lf\right\|_{\mathcal{V}}^2 + \beta\mathcal{R}(k)$$

- T is based on TLS method, measures the discrepancy on both data and operator;
- $L : \mathcal{V} \to \mathcal{V}$ is a linear bounded operator;
- α, β are the regularization parameters and γ is a scaling parameter;
- **double regularization** [You and Kaveh, 1996], $\mathcal{R} : U \to [0, +\infty]$ is proper **convex** function and **w-lsc**.

Theoretical results

DBL-RTLS is a regularization strategy:

- existence
- stability
- convergence
- convergence rates (New)

More info:
www.dk-compmath.jku.at/people/ibleyer

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Optimality condition

If the pair $(\bar{k}, \bar{f})$ is a minimizer of $J(k, f)$, then $(0,0) \in \partial J\left(\bar{k}, \bar{f}\right)$.

Theorem

Let $J : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ *be a nonconvex functional,*

$$J(u,v) = \varphi(u) + Q(u,v) + \psi(v)$$

where Q *is a nonlinear differentiable term and* φ, ψ *are lsc convex functions. Then*

$$\begin{aligned} \partial J(u,v) &= \{\partial\varphi(u) + D_u Q(u,v)\} \times \{\partial\psi(v) + D_v Q(u,v)\} \\ &= \{\partial_u J(u,v)\} \times \{\partial_v J(u,v)\} \end{aligned}$$

Remark:

- is difficult to solve wrt both (k, f)
- J is bilinear and biconvex (linear and convex to each one)
- applied **alternating minimization** method.

Alternating minimization algorithm

Require: $g_\delta, k_\epsilon, L, \gamma, \alpha, \beta$

1: $n = 0$
2: **repeat**
3: $f^{n+1} \in \arg\min_f J(k, f | k^n)$
4: $k^{n+1} \in \arg\min_k J(k, f | f^{n+1})$
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Proposition

The sequence generated by the function $J(k^n, f^n)$ *is non-increasing,*

$$J(k^{n+1}, f^{n+1}) \leq J(k^n, f^{n+1}) \leq J(k^n, f^n).$$

Assumptions:

(A1) B is strongly continuous, ie., if $(k^n, f^n) \rightharpoonup (\bar{k}, \bar{f})$ then $B(k^n, f^n) \to B(\bar{k}, \bar{f})$

(A2) B is weakly sequentially closed, ie., if $(k^n, f^n) \rightharpoonup (\bar{k}, \bar{f})$ and $B(k^n, f^n) \rightharpoonup g$ then $B(\bar{k}, \bar{f}) = g$

(A3) the adjoint of B' is strongly continuous, ie., if $(k^n, f^n) \rightharpoonup (\bar{k}, \bar{f})$ then $B'(k^n, f^n)^* z \to B'(\bar{k}, \bar{f})^* z$, $\forall z \in \mathscr{D}(B')$

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Theorem

Given regularization parameters $0 < \underline{\alpha} \leq \alpha$ *and* β*, compute AM algorithm. The sequence* $\{(k^{n+1}, f^{n+1})\}_{n+1}$ *has a weakly convergent subsequence, namely* $(k^{n_j+1}, f^{n_j+1}) \rightharpoonup (\bar{k}, \bar{f})$ *and the limit has the property*

$$J(\bar{k}, \bar{f}) \leq J(\bar{k}, f) \quad \text{and} \quad J(\bar{k}, \bar{f}) \leq J(k, \bar{f})$$

for all $f \in \mathcal{V}$ *and for all* $k \in \mathcal{U}$.

Proposition

Let $\{(k^n, f^n)\}_n$ *be a weakly convergent sequence generated by AM algorithm, where* $k^n \rightharpoonup \bar{k}$ *and* $f^n \rightharpoonup \bar{f}$*. Then there exists a subsequence* $\{k^{n_j}\}_{n_j}$ *such that* $k^{n_j} \to \bar{k}$ *and there exists* $\{\xi_k^{n_j}\}_{n_j}$ *with* $\xi_k^{n_j} \in \partial_k J(k^{n_j}, f^{n_j})$ *such that* $\xi_k^{n_j} \to 0$.

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Proposition

Let $\{n\}$ *be a subsequence of* $\mathbb{N}$ *such that the sequence* $\{(k^n, f^n)\}_n$ *generated by AM algorithm satisfies* $k^n \to \bar{k}$ *and* $f^n \rightharpoonup \bar{f}$*. Then* $f^{n_j} \to \bar{f}$ *and there exists* $\{\xi_f^{n_j}\}_{n_j}$ *with* $\xi_f^{n_j} \in \partial_f J(k^{n_j}, f^{n_j})$ *such that* $\xi_f^{n_j} \to 0$*.*

Remark: Graph of subdifferential mapping is sw-closed, ie., if $v_n \to \bar{v}$ and $\xi_n \rightharpoonup \bar{\xi}$ with $\xi_n \in \partial\varphi\,(v_n)$, then $\bar{\xi} \in \partial\varphi\,(\bar{v})$.

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Let $\{(k^n, f^n)\}_n$ *be the sequence generated by the AM algorithm, then there exists a subsequence converging towards to a critical point of* J*, ie.,*

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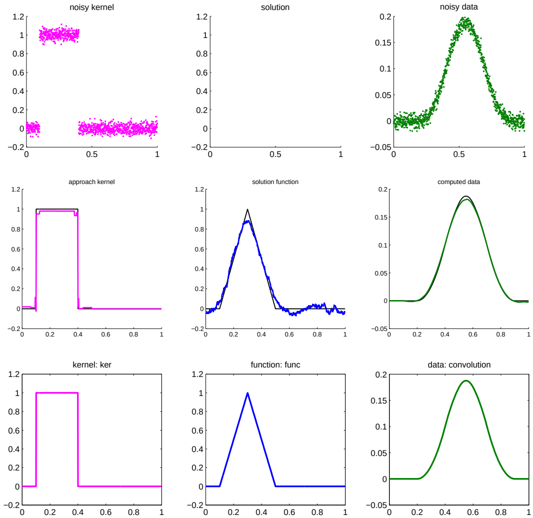
Overview

First numerical result

Convolution in 1D

$$\int_{\Omega} k(s-t)f(t)dt = g(s)$$

- **characteristic** kernel and **hat** function;
- space: $\Omega = [0,1]$, discretization: $N = 2048$ points;
- $\mathcal{R}(k) = \|k\|_{w,p}$ with $p = 1$
- Haar wavelet for $\{\phi\}_\lambda$ and $J = 10$;
- initial guess: $k^0 = k_\epsilon$, $\tau = 1.0$;
  - 1st. relative error: 10% and 10%.
  - 2nd. relative error: 0.1% and 0.1%.


noisy kernel
solution
noisy data
approach kernel
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kernel: ker
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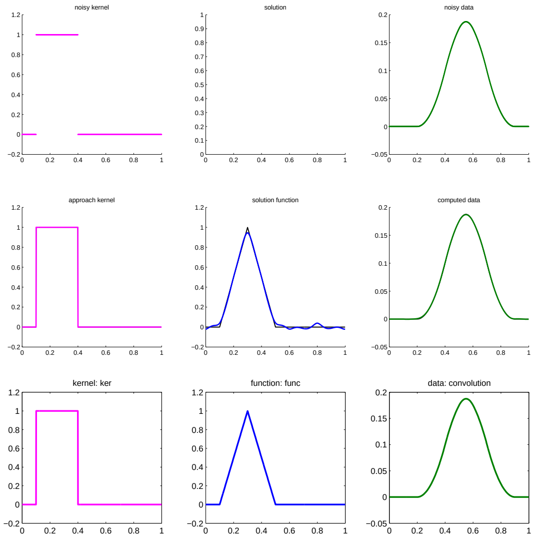
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−0.2 0 0.2 0.4 0.6 0.8 1 1.2
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
−0.05 0 0.05 0.1 0.15 0.2

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