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An identity on multinomial coefficients

Jiayue Qi

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JOHANNES KEPLER UNIVERSITY LINZ

LAND OBERÖSTERREICH

Motivation

The identity we focus on in this poster showed up when we studied the integral value of monomials in the Chow ring of stable curves of genus zero. The algorithm we provide to calculate that integral value is called "the forest algorithm" [1]. We need this identity to prove the correctness of the forest algorithm, in its base case. In the base case, the corresponding graphical representation for the monomials are called "sun-like trees". We found out that there is an equivalence between this identity and the integral value of sun-like trees. For more details on the background, motivation, or context, see [2].

Identity

For any r-many positive-integer parameters $m_1, m_2, \dots, m_r$, define $s_r := \sum_{i=1}^r m_i$. Denote a set of r-many indeterminates as $X_r := \{x_1, x_2, \dots, x_r\}$. Define $T_r := \{B \mid B \subset X_r, x_1 \in B\}$ and $\mathcal{B}_r := \{(B_1, B_2) \mid B_1 \in T_r, B_2 = X_r \backslash B_1\}$. Define $g_r : X_r \to \{m_1 - 1, m_2, \dots, m_r\}$ by $g_r(x_i) := m_1 - 1$ if $i = 1$ and $g_r(x_i) := m_i$ otherwise. Here note that the set $\{m_1 - 1, m_2, \dots, m_r\}$ may be a multi set in form but we just consider it as a normal set. For the convenience in the later writing, we introduce the following notation. Define for $B \subset X_r$, $S(B) := \sum_{x \in B} g_r(x)$ and $\binom{S(B)}{B} := \frac{S(B)!}{\prod_{x \in B}(g_r(x)!)}$. Based on the above preparation, the identity we want to prove can be formulated as follows.

Theorem 1

$$\binom{s_r}{m_1, m_2, \dots, m_r} = \sum_{(B_1, B_2) \in \mathcal{B}_r} \binom{s_r - r + 1}{S(B_2) - |B_2|} \binom{S(B_1)}{B_1} \binom{S(B_2)}{B_2},$$

where $|B_2|$ refers to the cardinality of B_2.

Proof

Basic idea of the proof

The right-hand side is considered as the number of partitions of the set $\{1, \dots, s_r\}$ into r parts of sizes $m_1, \dots, m_r$ respectively, while the left-hand side will be interpreted as the sum of the cardinalities of pre-images of some mapping φ_r.

Define $S_r := \{(P_1, P_2, \dots, P_r) \mid \cup_{i=1}^r P_i = \{1, 2, \dots, s_r\}, |P_i| = m_i\}$ and $L_r := \{2, 3, \dots, r\}$. For $A \subset \{1, 2, \dots, r\}$, define $P_A := \cup_{i \in A} P_i$ and $X_A := \{x_i \mid i \in A\}$. Define $\varphi_r : S_r \to T_r$, $(P_1, \dots, P_r) \mapsto B$ by the following φ_r algorithm.

the function φ_r

- **Input:** $(P_1, \dots, P_r) \in S_r$.
- **Output:** $B \in T_r$.

$B \leftarrow \{x_1\}$.
$A \leftarrow L_r \cap P_1$.
While $A \neq \emptyset$: $B := B \cup X_A$, $A := L_r \cap P_A$.
Return B.

Function φ_r is a well-defined function; one can verify that it is a surjection. Then using a basic property of any mapping, we obtain $\cup_{B \in T_r} \varphi_r^{-1}(B) = S_r$ and $|S_r| = \sum_{(B, X_r \backslash B) \in \mathcal{B}_r} |\varphi_r^{-1}(B)| = \sum_{B \in T_r} |\varphi_r^{-1}(B)|$. Now we only need to show one thing, namely the following lemma.

Lemma 2

For any $B_1 \in T_r$, define $B_2 := X_r \setminus B_1$, then

$$|\{x \in S_r \mid \varphi_r(x) = B_1\}| = \binom{s_r - r + 1}{S(B_2) - |B_2|} \binom{S(B_1)}{B_1} \binom{S(B_2)}{B_2}.$$

In order to prove the Lemma 2, we need the following proposition.

Proposition 3

If $\varphi_r(P_1, \dots, P_r) = B_1$ for some $(P_1, \dots, P_r) \in S_r$ and $B_1 \in T_r$; denote $B_2 := X_r \setminus B_1$. Then $P_{F_{B_1}} \cap L_r = F_{B_1} \setminus \{1\}$, where $F_B := \{i \mid x_i \in B\}$. Consequently, we have $P_{F_{B_2}} \cap L_r = F_{B_2}$ and $|P_{F_{B_2}} \cap L_r| = |B_2|$.

Inspired by the above proposition, in order to find the configuration in S_r such that it is mapped to a given $B \in T_r$ by the function φ_r, we can instead think of this problem in the following way.

First divide $K_r := \{1, \dots, s_r\}$ into two subsets: one is P_{F_B}, the other is $K_r \setminus P_{F_B}$. Then by Proposition 3, the division of elements in L_r is fixed, hence for this step we have some freedom to devide elements in $K_r \setminus L_r$ into two groups. This is a fundamental problem in combinatorics. We need to choose $|P_{F_{B_2}}| - |B_2| = S(B_2) - |B_2|$ many elements from $|K_r \setminus L_r| = s_r - |L_r| = s_r - r + 1$ many elements. There are $\binom{s_r - r + 1}{S(B_2) - |B_2|}$ many ways to do so. Then we need to find what should be the configurations within each of the two groups. Consider the definition of φ_r, we see that no matter how we arrange the elements in $P_{F_{B_2}}$, the value of φ_r is not influenced. Therefore, there are $\binom{S(B_2)}{B_2}$ many configurations for the elements in $P_{F_{B_2}}$. As for the arrangements in $P_{F_{B_1}}$, they need to obey certain rules in order to guarantee that the image under φ_r is B_1.

From the analysis above, the first and the third coefficients on the right hand side of the equation in Lemma 2 are both explained in a combinatorial way. In order to prove the lemma, we only need to prove that given $B_1 \in T_r$, the number of configurations for the elements in $P_{F_{B_1}}$ is exactly $\binom{S(B_1)}{B_1}$. Recall the definition of $\binom{S(B_1)}{B_1}$, one can see that the remaining work for the proof of Lemma 2 is equivalent to proving the following result.

Proposition 4

Recall that $s_k := \sum_{i=1}^k m_i$ and that $X_k := \{x_1, \dots, x_k\}$. Then we have

$$f_k(m_1, m_2, \dots, m_k) = \binom{s_k - 1}{m_1 - 1, m_2, \dots, m_k}, \; k \in \mathbb{N}^+, \; m_i \in \mathbb{N}^+,$$

where $f_k : (\mathbb{N}^+)^k \to \mathbb{N}$,

$$(m_1, m_2, \dots, m_k) \mapsto |\{(P_1, P_2, \dots, P_k) \in S_k \mid |P_i| = m_i, \; \varphi_k(P_1, P_2, \dots, P_k) = X_k\}|.$$

To explain the above defined function f_k in another way, we have $f_k(m_1, \dots, m_k) = \#\{\varphi_k^{-1}(X_k) \mid |P_i| = m_i\}$; it is the cardinality of the fiber of φ_k given that $|P_i| = m_i$. To sum up the story line so far, in order to prove Theorem 1, we only need to prove Proposition 4. The proof of Proposition 4 is done by a double-layered induction, also with the help of the following known identity on multinomial coefficients.

Lemma 5

For all $s, k, m_1, \dots, m_k \in \mathbb{N}$ with $m_1 + \dots + m_k = s$, $s \geq 1$ and $k \geq 2$, we have $\binom{s}{m_1, \dots, m_k} = \sum_{i=1}^k \binom{s-1}{m_1, \dots, m_i - 1, \dots, m_k}$.

Example

Example

In order to check through the identity with an example, we only need to fix $r \in \mathbb{N}^+$ and $m_1, \dots, m_r$. Take $r = 3$, $m_1 = 2$, $m_2 = 3$, $m_3 = 3$, for instance. Then $s = \sum_{i=1}^3 m_i = 2 + 3 + 3 = 8$. Hence LHS $= \binom{s}{m_1, m_2, m_3} = \binom{8}{2,3,3} = 560$. Recall the identity, we have $\mathcal{B} = \{(\{x_1\}, \{x_2, x_3\}), (\{x_1, x_2\}, \{x_3\}), (\{x_1, x_3\}, \{x_2\}), (\{x_1, x_2, x_3\}, \emptyset)\}$. Now we need to go through these four elements of $\mathcal{B}$, starting from $B_1 = \{x_1\}$, $B_2 = \{x_2, x_3\}$. Then we have $S(B_1) = m_1 - 1 = 2 - 1 = 1$, $S(B_2) = m_2 + m_3 = 3 + 3 = 6$, $\binom{S(B_1)}{B_1} = \frac{S(B_1)!}{g(x_1)!} = \frac{1}{1} = 1$, and $\binom{S(B_2)}{B_2} = \binom{6}{3,3} = 20$,

$$\binom{s - r + 1}{S(B_2) - |B_2|} = \binom{8 - 3 + 1}{6 - 2} = \binom{6}{4} = 15.$$

So the corresponding summand for $(\{x_1\}, \{x_2, x_3\})$ is

$$\binom{s - r + 1}{S(B_2) - |B_2|} \binom{S(B_1)}{B_1} \binom{S(B_2)}{B_2} = 15 \cdot 1 \cdot 20 = 300.$$

Going through the similar process, we obtian the other three summands on RHS as: $\binom{6}{2} \cdot \binom{4}{1,3} = 60$, $\binom{6}{2} \cdot \binom{4}{1,3} = 60$, and $\binom{7}{1,3,3} = 140$. Summing up the four summands: $300 + 60 + 60 + 140 = 560$. The identity is verified for this example.

Acknowledgements

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References

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[2] J. Qi. A tree-based algorithm on monomials in the Chow group of zero cycles in the moduli space of stable pointed curves of genus zero. arXiv:2101.03789.