Moduli space of n marked points on a projective line

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Motivation

The focus of this poster is the compactification of the moduli space of n pairwise distinct points on a projective line which is a smooth projective variety of dimension $n-3$. It has been constructed by Knudsen and Mumford. Their construction has been used for theoretical physics, resolution of singularities, and kinematics. It has been called "the main tool of modern enumerative geomety". However, their construction is rather long and complicated. We give a self-contained construction of a variety which is isomorphic to the Knudsen-Mumford moduli space, using only basic algebraic geometry. We will not go into details of their construction.

Basics

We introduce the abbreviations ∞, $\mathbf{0}$, $\mathbf{1}$ for the three points $(1:0),(0:1),(1:1) \in \mathbb{P}^1$. Given a quadruple $(p_1,p_2,p_3,p_4) \in (\mathbb{P}^1)^4$. If the four points are pairwise distinct, its **cross ratio** is defined to be $((p_1-p_3)(p_2-p_4):(p_1-p_4)(p_2-p_3))$. Later we use the notation $\gamma_q(p)$, where $p \in (\mathbb{P}^1)^n$ and q a quadruple of four entries, to define the cross ratio of these four entries on p. If ∞ is contained in one of the four entries, we just remove the two factors that contains infinity from the above formula. When the four places are pairwise distinct, it is not hard to check that the cross ratio is different from ∞, $\mathbf{0}$, or $\mathbf{1}$. In other cases, the definition is as follows: $p_1=p_2$ or $p_3=p_4$ iff $\gamma(p_1,p_2,p_3,p_4)=\mathbf{1}$; $p_1=p_3$ or $p_2=p_4$ iff $\gamma(p_1,p_2,p_3,p_4)=\mathbf{0}$; $p_1=p_4$ or $p_2=p_3$ iff $\gamma(p_1,p_2,p_3,p_4)=\infty$; If three or four points coincide in the quadruple, then the cross ratio is not defined.

Now we come to some basic settings. Let $n \geq 3$ be an integer, we study the equivalence induced by the group action of $PGL(2,\mathbb{C})$ on $(\mathbb{P}^1)^n$ — an n-tuple $(p_1,...,p_n) \in (\mathbb{P}^1)^n$ is called an **n-gon**. we abbreviate this group as PGL_2 from now on. Elements in PGL_2 are all the 2×2 matrices which has non-zero determinant. Two n-gons are equivalent if there is a projective linear transformation transforming one into the other. In our setting this transformation is nothing more than the Möbius transformation. A *Möbius transformation* of the complex plane is a rational function of the form $f(z)=\frac{az+b}{cz+d}$ of one complex variable z; a,b,c,d here are complex numbers satisfying $ad-bc \neq 0$. Group PGL_2 acts on $(p_1,...,p_n)$ by $(p_1,...,p_n)^\sigma := (p_1^\sigma,...,p_n^\sigma)$ for all $\sigma \in PGL_2$. The equivalent classes are called **orbits**. Note that the group PGL_2 has the property of 3-sharp-transitivity, meaning that there is a unique group element which transfers the three pairwise distinct points to another three pairwise distinct points. The n-gon is **dromedary** if all points are pairwise distinct. Two dromedary n-gons are equivalent if and only if all corresponding cross ratios of all quadruples coincide. Because of the 3-sharp-transitivity of PGL_2, we can fix three points at the beginning. Hence, dromedary orbits are in bijective correspondence with the points in U_n which is defined as the open subset of all points $(c_4,\dots,c_n) \in (\mathbb{P}^1)^{n-3}$ where $c_i \notin \{\infty,\mathbf{0},\mathbf{1}\}$ for $i \in \{4,\dots,n\}$ and $c_i \neq c_j$ if $i \neq j$, where $i,j \in \{4,\dots,n\}$. Then, the equivalence classes are in bijective correspondence with the points of an open subset $(\mathbb{P}^1)^{n-3}$ parametrized by $n-3$ cross ratios.

The open set U_n **is the moduli space of** n **distinct points on** $\mathbb{P}^1$, under PGL_2 group action. It is an open subset of $(\mathbb{P}^1)^{n-3}$, and $(\mathbb{P}^1)^{n-3}$ is indeed a compactification of it, which is projective and smooth. However, the three prefixed entries are somehow special, so it is not symmetric under random permutation of the labels. **We want to find a "good" compactification of U_n which is smooth, projective, and symmetric under permutation of labels.** Basically we need to construct a compactification for U_n somehow basing the consideration on those orbits that are not dromedary. We managed to find it! We denote it by $\mathcal{M}_n$.

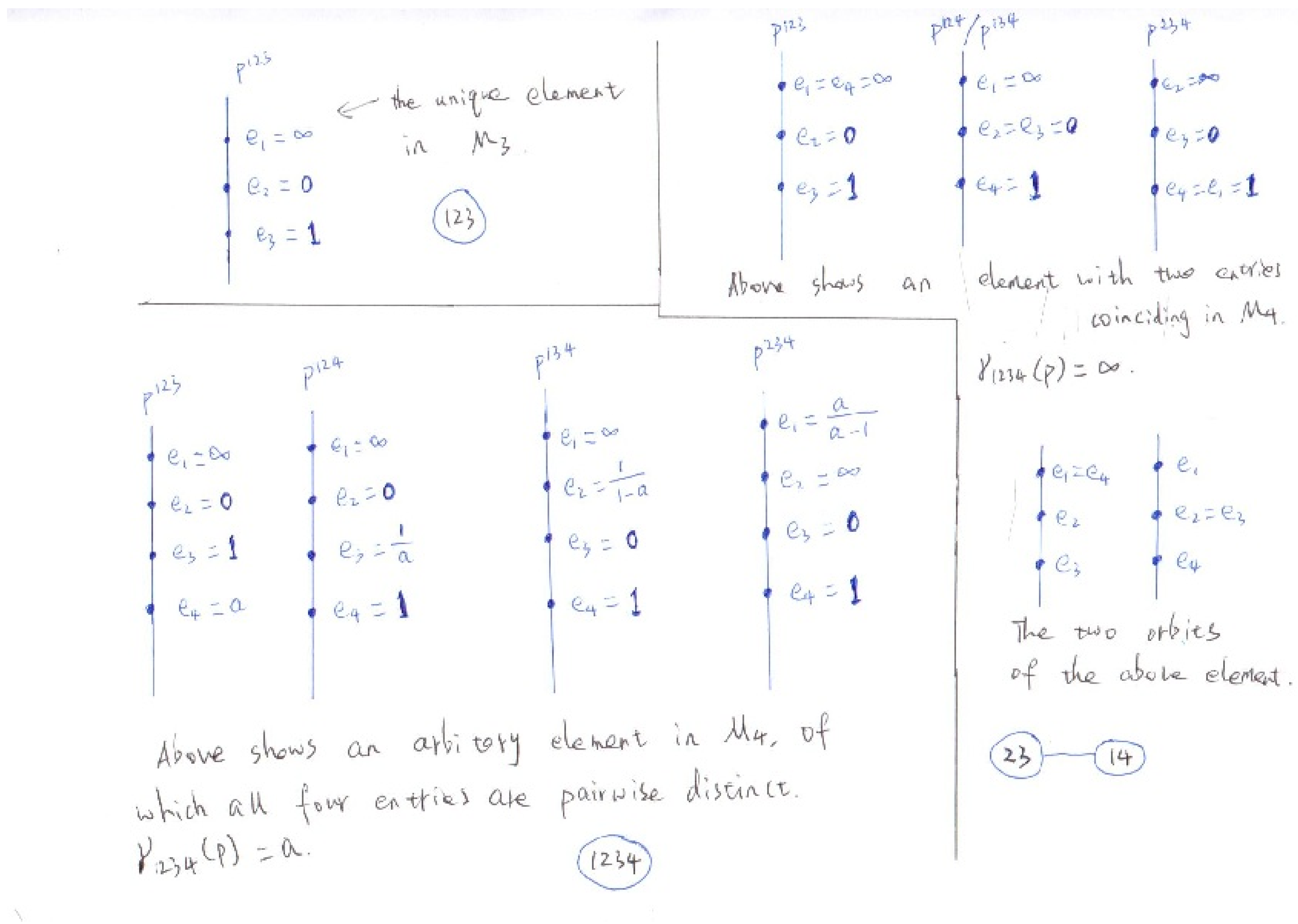
Moduli space

Denote by $T_n := \{(i,j,k) \mid i,j,k \in \{1,...,n\}, i<j<k\}$. Sometimes we use short notation for the elements in T_n, for instance, 123 represents $\{1,2,3\}$,etc. Let $\mathcal{M}_n := \{p \in ((\mathbb{P}^1)^n)^{T_n} \mid \forall t=(i,j,k) \in T_n : p_i^t=\infty, p_j^t=\mathbf{0}, p_k^t=\mathbf{1}, \forall t_1,t_2 \in T_n, \forall q \in Q : \gamma_q(p^{t_1})=\gamma_q(p^{t_2})$ if both sides are defined$\}$. Note that we define $\mathcal{M}_n$ only for $n \geq 3$, otherwise there is no triple to consider. Let's see some examples, so as to understand better the definition.

Example
The set $\mathcal{M}_3$ consists of only one element which can be denoted as p. The element p contains only one 3-gon: $p^{(1,2,3)}$. We have $p_1^{(1,2,3)}=\infty$, $p_2^{(1,2,3)}=\mathbf{0}$, $p_3^{(1,2,3)}=\mathbf{1}$. Since the number of entries is not enough to talk about cross ratios, with this we finish the exploration of $\mathcal{M}_3$. On the left upper corner of the figure, we see the graphical illustration of $\mathcal{M}_3$.

Example
The set $\mathcal{M}_4$ consists of infinitely many elements. Each one of them contains four elements: p^{123}, p^{124}, p^{134}, p^{234}. Denote any element in $\mathcal{M}_4$ by p. When four entries of p are pairwise distinct, we have that $p_1^{123}=\infty$, $p_2^{123}=\mathbf{0}$, $p_3^{123}=\mathbf{1}$, assume w.l.o.g. that $p_4^{123}=a \in \mathbb{P}^1 \setminus \{\infty,\mathbf{0},\mathbf{1}\}$. With the requirement on cross ratios in the definition of $\mathcal{M}_n$, we can calculate out precisely the other three 4-gons. Since $\gamma_{1234}(p^{123})=\gamma_{1234}(p^{124})$, we know that $p_3^{124}=\frac{1}{a}$. Analogously, we obtain that $p_2^{134}=\frac{1}{1-a}$ and $p_1^{234}=\frac{a}{a-1}$. On the left bottom of the figure, we see the graphical illustration of such elements in $\mathcal{M}_4$.

Since we only discuss here the situation when $n \geq 3$, there should be at least three pairwise distinct entries for the 4-gons. So the only case that is left is when two entries coincide. There are in total three elements in M_4 in this case. First one is $p_1^{123}=p_4^{123}$. Then by the requirement of cross ratios being equal, we deduce that $p_2^{124}=p_3^{124}$, $p^{124}=p^{134}$ and $p_4^{234}=p_1^{234}$. Second one is $e_2=e_4$ on p^{123} and p^{134}, $e_1=e_3$ on p^{124} and p^{234}. Third one is $e_3=e_4$ on p^{123} and p^{124}, $e_1=e_2$ on p^{134} and p^{234}. We show the first one in a graphical way on the right upper corner of the figure.



Loaded graph

Let $x \in \mathcal{M}_n$. If p is an n-gon of x, then a subset $I \subset N$ is called a **cluster** of p or of its orbit (under PGL_2 action) $[p]$, iff $\forall i,j \in I, k \in N \setminus I$ we have $p_i=p_j \neq p_k$. For each $x \in M_n$, we define a graph (V,E) as follows. V is the set of all PGL_2-orbits of n-gons of x. There is an edge between $[p]$ and $[q]$ iff $[p]$ has a cluster I, $[q]$ has a cluster J and (I,J) is a bi-partition of N. For each vertex v, $H(v)$ is the set of labels i such that $\{i\}$ is a cluster of v. Elements in $H(v)$ are called the **singletons** of v. The graph (V,E), together with the sets $H(v)$ for $v \in V$, is called the **loaded graph** of x and denoted by $L(x)$. Please find the corresponding loaded graphs next to the element respectively, in the figure.

Lemma
For any $x \in M_n$, the loaded graph of x is a tree.

A **loaded tree with labeling set** N is a tree (V,E) together with a collection $(H(v))_{v\in V}$ of subsets of N so that its non-empty elements form a partition of N, and that $|H(v)|+\deg(v) \geq 3$ for each vertex v.

Theorem
Let (V,G,H) be the loaded graph of $x \in M_n$. Then (V,G,H) is a loaded tree with n labels. Let (V,G,H) be a loaded tree with n nodes. Then there exists a point $x \in M_n$ such that $L(x)=(V,G,H)$.

With the help of these combinatorial structures, we can prove the following result.

Theorem
The variety M_n is smooth and of dimension $n-3$.