Topology analysis of complex curves singularities using knot theory

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Motivation

Why study the topology of a complex curve singularity?
What is the topology of a singularity?
How to compute the topology?
Why using knot theory?

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☞ To compute the genus of plane complex curves!

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☛ To compute the genus of plane complex curves!
☞ The algebraic link of the singularity!

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Motivation

Why study the topology of a complex curve singularity?
What is the topology of a singularity?
How to compute the topology?
Why using knot theory?

- ☛ To compute the genus of plane complex curves! We use
- ☛ The algebraic link of the singularity!
- ☞ We propose a symbolic-numeric algorithm for this purpose!

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Motivation

Why study the topology of a complex curve singularity?
What is the topology of a singularity?
How to compute the topology?
Why using knot theory?

- ☛ To compute the genus of plane complex curves!
- ☛ The algebraic link of the singularity!
- ☛ We propose a symbolic-numeric algorithm for this purpose!
- ☞ The proposed algorithm is stable w.r.t. small perturbations!

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Motivation

Why is this proposed symbolic-numeric algorithm "special"?

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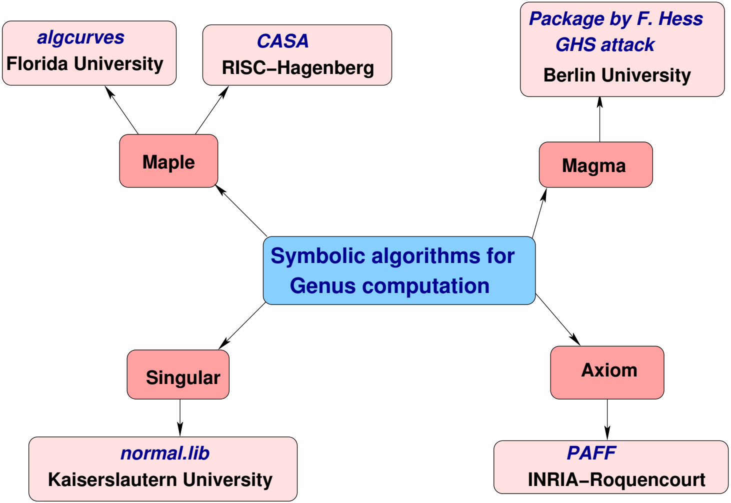
Motivation

At present, there exists several...



Motivation

But...

For moderate symbolic input data: they are too expensive

↑

Disadvantages of Symbolic algorithms for Genus computation

↓

For numeric input data: they are unusable

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Motivation

For instance, in Maple using algcurves package...

```
> with(algcurves);
[AbelMap, Siegel, Weierstrassform, algfun_series_sol, differentials, genus,
    homogeneous, homology, implicitize, integral_basis, is_hyperelliptic,
    j_invariant, monodromy, parametrization, periodmatrix, plot_knot,
    plot_real_curve, puiseux, singularities]
```

> $f := x^2\, y + y^4$

$$f := x^2\, y + y^4$$

> $genus(f, x, y)$

$$-1$$

> $g := 1.02 \cdot x^2\, y + 1.12 \cdot y^4$

$$g := 1.02\, x^2\, y + 1.12\, y^4$$

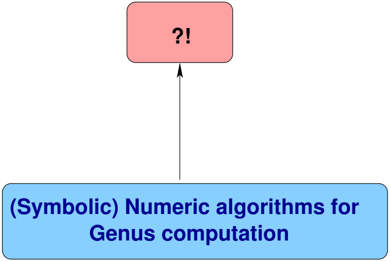
> $genus(g, x, y)$

```
Error, (in content/polynom) general case of floats not handled
```

>

Motivation

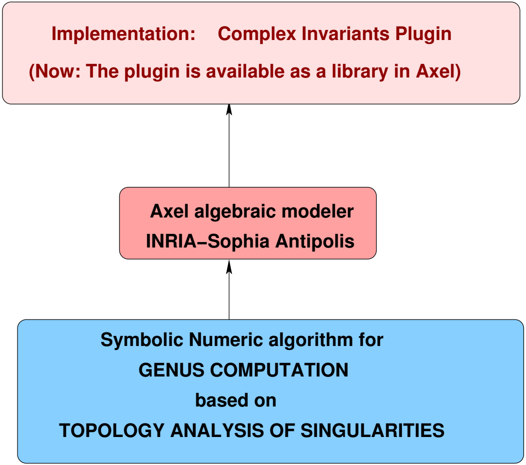
Thus we need...



Motivation

Hopefully...

Project: Symbolic-Numeric techniques for genus computation (initiated by J. Schicho).



Other numeric method was reported (in the group of R. Sendra).

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What?

- Input:
  - $F \in \mathbb{C}[x, y]$ squarefree with coefficients of limited accuracy:
  - $C = \{(x, y) \in \mathbb{C}^2 | F(x, y) = 0\}$ complex algebraic curve of degree m.
  - $\epsilon \in \mathbb{R}_+^*$ a non-zero positive real number, the input parameter.
- Output:
  - the algebraic link/topology of each singularity $s \in Sing(C)$, where $Sing(C)$ is the set of singularities of the curve C.

What?

- Input:
  - $F \in \mathbb{C}[x, y]$ squarefree with coefficients of limited accuracy:
    - integers or rational numbers: $1, -2, \dfrac{1}{2}$.
    - or real numbers. For 1.001 we associate a tolerance of $\sigma = 10^{-3}$.
  - $C = \{(x, y) \in \mathbb{C}^2 | F(x, y) = 0\}$ complex algebraic curve of degree m.
  - $\epsilon \in \mathbb{R}_+^*$ a non-zero positive real number, the input parameter.
- Output:
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How?

- Strategy for computing the topology of all the singularities of the curve

$$\boxed{\text{Plane complex algebraic curve}}$$

$$\Big\downarrow \text{compute} \;\text{numerically}$$

$$\boxed{\text{Singularities}}$$

$$\Big\downarrow \text{compute} \;\text{numerically-symbolically}$$

$$\boxed{\text{Algebraic link and operations on it } (\epsilon \text{ needed})}$$

$$\Big\downarrow \text{compute} \;\text{symbolically}$$

$$\boxed{\text{Alexander polynomial, } \delta\text{-invariant, GENUS}}$$

How?

- Strategy for computing the topology of all the singularities of the curve

$$\begin{array}{c}
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\text{compute} \Big\downarrow \text{numerically} \\
\boxed{\text{Singularities moved in origin}} \\
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\end{array}$$

Solving the problem
Implementation of the algorithm

- *Axel* algebraic geometric modeler [a]

[a] Acknowledgements: Julien Wintz

Solving the problem
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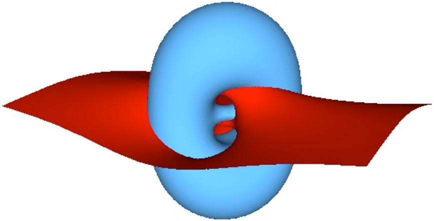
  - provides algebraic tools for:
    - implicit surfaces;



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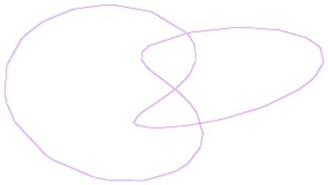

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Solving the problem

Implementation of the algorithm

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  - provides algebraic tools for:
    - implicit surfaces;
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  - free, available at:

http://axel.inria.fr/

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First

$$\begin{tikzcd}
\boxed{\text{Plane complex algebraic curve}} \arrow[d, "\text{compute}"', "\text{numerically}"] \\
\boxed{\text{Singularities}} \arrow[d, "\text{compute}"', "\text{numerically-symbolically}"] \\
\boxed{\text{Algebraic link and operations on it } (\epsilon \text{ needed})} \arrow[d, dashed, "\text{compute}"', "\text{symbolically}"] \\
\boxed{\text{Alexander polynomial, } \delta\text{-invariant, GENUS}}
\end{tikzcd}$$

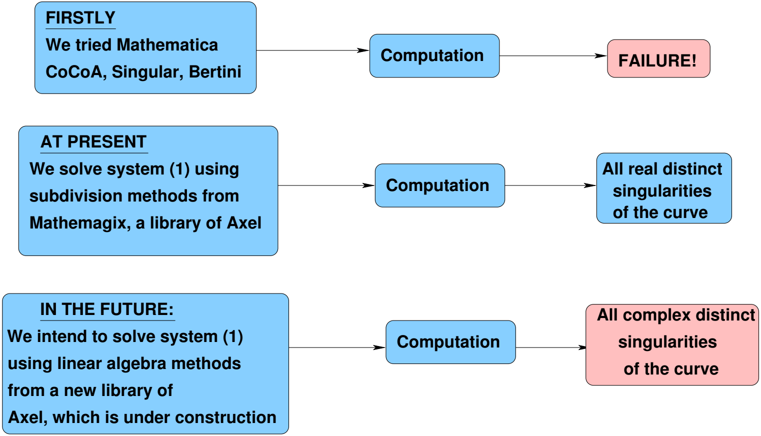
Computing the singularities of the curve

- Input:
  - $F(x,y) \in \mathbb{C}[x,y]$ squarefree with coefficients of limited accuracy.
  - $C = \{(x,y) \in \mathbb{C}^2 | F(x,y) = 0\}$ complex algebraic curve of degree m.
- Output:
  - $Sing(C) = \{(x_0,y_0) \in \mathbb{C}^2 | F(x_0,y_0) = 0, \frac{\partial F}{\partial x}(x_0,y_0) = 0, \frac{\partial F}{\partial y}(x_0,y_0) = 0\}$
- Method: We solve the overderminate system of polynomial equations with coefficients of limited accuracy in $\mathbb{C}^2$:

$$\begin{cases} F(x_0,y_0) = 0 \\ \frac{\partial F}{\partial x}(x_0,y_0) = 0 \\ \frac{\partial F}{\partial y}(x_0,y_0) = 0 \end{cases}, \tag{1}$$

Computing the singularities of the curve

For input polynomials with coefficients of limited accuracy:



Note: We assume the subdivision methods return all the singularities.

Next

$$\begin{array}{c}
\boxed{\text{Plane complex algebraic curve}} \\
\text{compute} \Big\downarrow \text{numerically} \\
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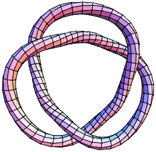
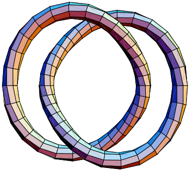
Knot theory - preliminaries

- A **knot** is a piecewise linear or a differentiable simple closed curve in $\mathbb{R}^3$.
- A **link** is a finite union of disjoint knots.
- Links resulted from the intersection of a given curve with the sphere are called **algebraic links**.

Trefoil Knot

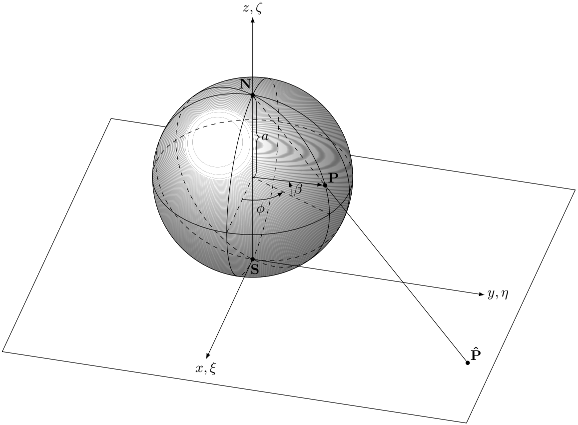


Hopf Link



Computing the algebraic link of the singularity

- Why the algebraic link of a singularity?
  - helps to study the topology of a complex curve near a singularity;
- How do we compute the algebraic link?
  - use the generalization of the stereographic projection;



Computing the link of the singularity

Method (based on Milnor's results)

1. Let $C = \{(a,b,c,d) \in \mathbb{R}^4 | F(a,b,c,d) = 0\}$ s.t. $(0,0,0,0) \in Sing(C)$
2. Consider $S_{(0,\epsilon)} := S = \{(a,b,c,d) \in \mathbb{R}^4 | a^2 + b^2 + c^2 + d^2 = \epsilon^2\}$,
   $X = C \bigcap S_{(0,\epsilon)} \subset \mathbb{R}^4$
3. For $P \in S \setminus C$, $f : S \setminus \{P\} \to \mathbb{R}^3, (a,b,c,d) \mapsto (u = \frac{a}{\epsilon - d}, v = \frac{b}{\epsilon - d}, w = \frac{c}{\epsilon - d})$,
   $f^{-1} : \mathbb{R}^3 \to S \setminus \{P\}$
   $(u,v,w) \mapsto (a = \frac{2u\epsilon}{n}, b = \frac{2v\epsilon}{n}, c = \frac{2w\epsilon}{n}, d = \frac{\epsilon(u^2+v^2+w^2-1)}{n})$, where $n = 1 + u^2 + v^2 + w^2$.
4. Compute $f(X) = \{(u,v,w) \in \mathbb{R}^3 | F(\frac{2u\epsilon}{n}, \frac{2v\epsilon}{n}, ...) = 0\} \Leftrightarrow$
   $f(X) = \{(u,v,w) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$
   $f(X)$ is an implicitly defined algebraic curve!
   For small ϵ, $f(X)$ is a link (a differentiable algebraic link).

Computing the link of the singularity

We use Axel for implementation. Why Axel?

It is the only system to contain the implementation of a method for certified topology of smooth implicit curves in $\mathbb{R}^3$!

- For $C = \{(x,y) \in \mathbb{C}^2 | x^3 - y^2 = 0\} \subset \mathbb{R}^4$ we compute with the previous method in Axel:

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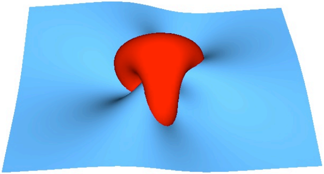
- $f(C \cap S) = f(X) := L =$
  $= \{(u,v,w) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$



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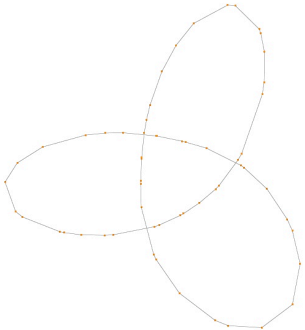
- $Graph(L) = \langle \mathcal{V}, \mathcal{E} \rangle$ with
  $\mathcal{V} = \{p = (m,n,q) \in \mathbb{R}^3\}$
  $\mathcal{E} = \{(i,j) | i,j \in \mathcal{V}\}$



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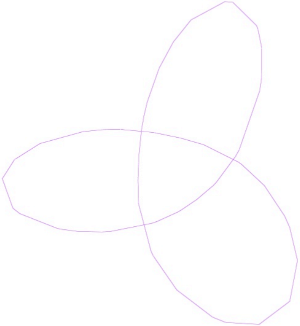
  $\mathcal{E} = \{(i, j) | i, j \in \mathcal{V}\}$
- s.t. $Graph(L) \cong_{isotopic} L$



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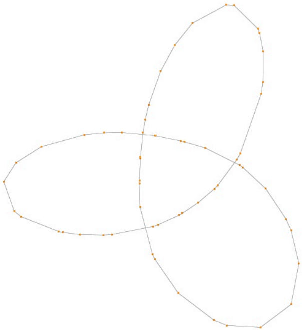
- For $C = \{(x, y) \in \mathbb{C}^2 | x^3 - y^2 = 0\} \subset \mathbb{R}^4$ we compute with the previous method in Axel:
- $f(C \cap S) = f(X) := L =$
  $= \{(u, v, w) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$
- $Graph(L) = \langle \mathcal{V}, \mathcal{E} \rangle$ with
  $\mathcal{V} = \{p = (m, n, q) \in \mathbb{R}^3\}$
  $\mathcal{E} = \{(i, j) | i, j \in \mathcal{V}\}$
- s.t. $Graph(L) \cong_{isotopic} L$
- $Graph(L)$ is the topology of L,
  a piecewise linear approximation
  for the differentiable algebraic link L;



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Computing the link of the singularity

We use Axel for the implementation. Why Axel?
It is the only system to contain the implementation of a method for certified topology of smooth implicit curves in $\mathbb{R}^3$!

- For $C = \{(x, y) \in \mathbb{C}^2 | x^3 - y^2 = 0\} \subset \mathbb{R}^4$

Computing the link of the singularity

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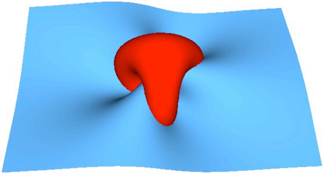
- For $C = \{(x, y) \in \mathbb{C}^2 | x^3 - y^2 = 0\} \subset \mathbb{R}^4$
- and $L =$
  $= \{(u, v, w) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$



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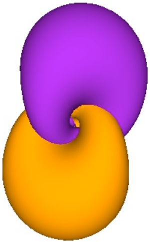
- and $L =$
  $= \{(u,v,w) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$
- we also compute (for visualization reasons)
  $S^{'} = \{(u,v,w) \in \mathbb{R}^3 | ReF(...) + ImF(...) = 0\}$
  $S^{''} = \{(u,v,w) \in \mathbb{R}^3 | Re(F) - ImF(...) = 0\}$



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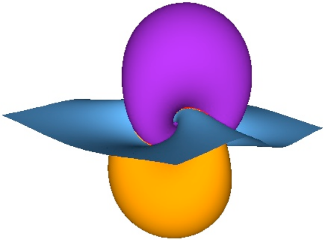
- we also compute (for visualization reasons)
  $S^{'} = \{(u, v, w) \in \mathbb{R}^3 | ReF(...) + ImF(...) = 0\}$
  $S^{''} = \{(u, v, w) \in \mathbb{R}^3 | Re(F) - ImF(...) = 0\}$
- L is the intersection of any 2 of the surfaces:
  $ReF(...), ImF(...)$
  $ReF(...) + ImF(...), ReF(...) - ImF(...)$



Next

Plane complex algebraic curve

↓ compute numerically

Singularities moved in origin

↓ compute numerically-symbolically

Algebraic link and operations on it (ϵ needed)

↓ compute symbolically

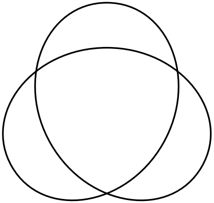
Alexander polynomial, δ-invariant, GENUS

Knot theory - preliminaries

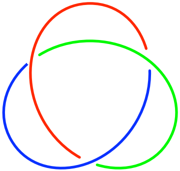
A knot projection is a **regular projection** if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.
A double point of a regular projection is a **crossing point**.

A **diagram** is the image under regular projection, together with the information on each crossing telling which branch goes over/under.
An **arc** is the part of a diagram between two undercrossings.

Regular projection



Diagram

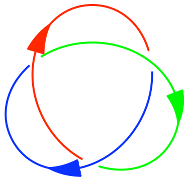


Knot theory - preliminaries

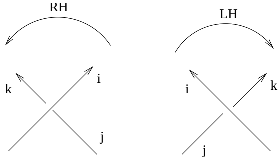
A diagram together with a given orientation of the link is called an **oriented diagram**.

A crossing is:
-**righthanded** if the underpass traffic goes from right to left.
-**lefthanded** if the underpass traffic goes from left to right.

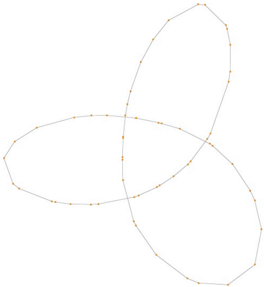
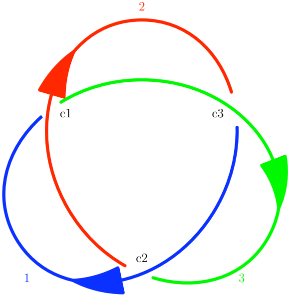
Oriented diagram



Crossings



Computing operations on the algebraic link

-  $\overset{?}{\Rightarrow}$ 

- $G(L) = \langle P, E \rangle$ $\qquad\qquad$ $D(L)$
- p(index,x,y,z)

  e(indexS, indexD)

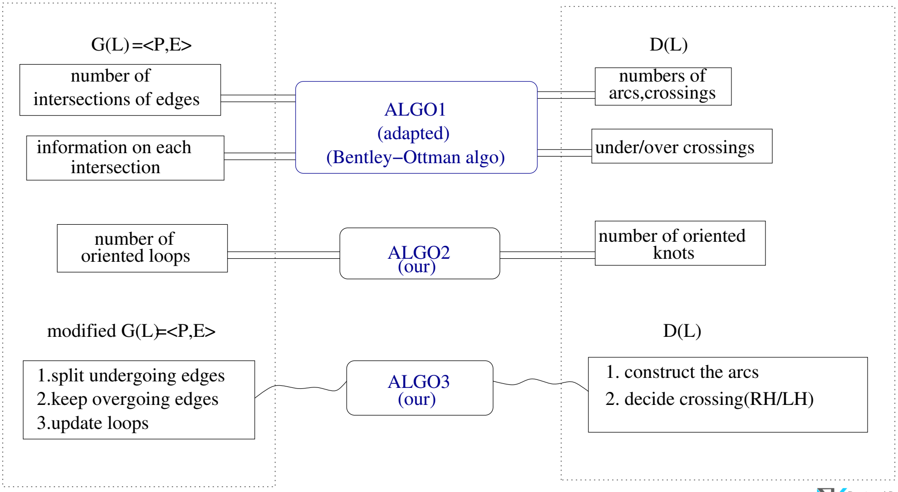
$\longrightarrow$ number of arcs, crossings

$\longrightarrow$ type of crossings (under, over)

$\longrightarrow$ number of knots in the link(orientation)

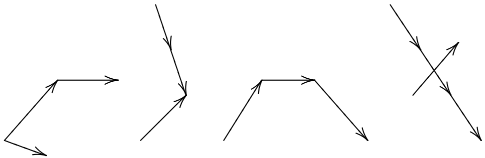
Computing operations on the algebraic link

By performing operations on $G(L)$ we obtain the elements of $D(L)$!



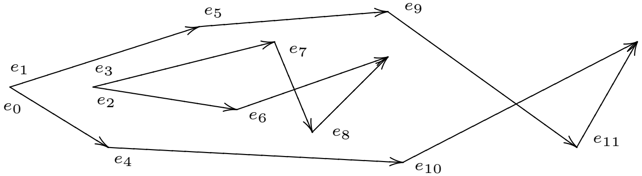
Algorithm 1 - Adapted version of Bentley-Ottman

- Input: S a set of "short" edges ordered from left to right:
  - A "short" edge is an edge whose projection contains at most one crossing point.

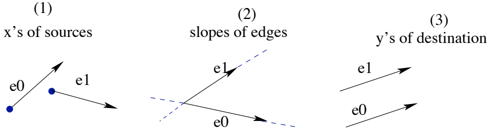


- Output: I - the set of all intersections among edges of S and
  - for each $p = e_i \cap e_j \in I$, the "arranged" pair of edges (e_i, e_j), i.e e_i is below e_j in $\mathbb{R}^3$

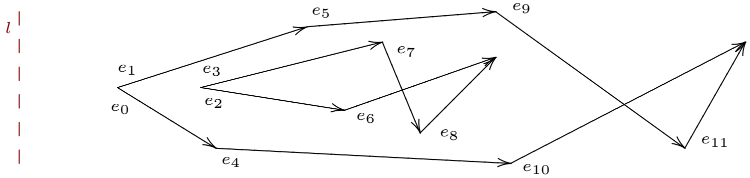
Algorithm 1 - Adapted version of Bentley-Ottman



- First: the edges are ordered by criteria (1),(2),(3):

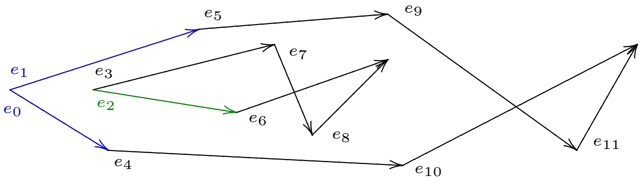


Algorithm 1 - Adapted version of Bentley-Ottman



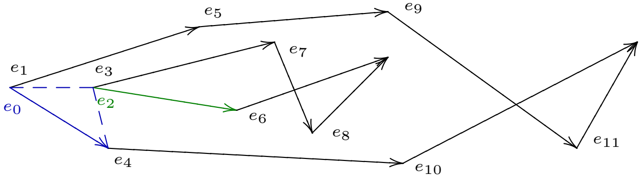
- we consider l a sweep line
- we keep track of two lists:
  $E = \{e_0, e_1, ..., e_{11}\}$ the list of ordered edges
  $Sw = \{?\}$ the list of event points
- while traversing E we insert the edges in Sw in the "right" position
- That is...

Algorithm 1 - Adapted version of Bentley-Ottman



- $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$
- $Sw = \{e_0, e_1\}$

Algorithm 1 - Adapted version of Bentley-Ottman



- $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$
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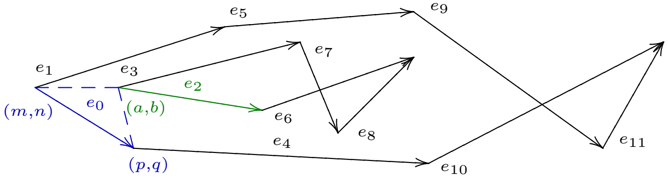
Algorithm 1 - Adapted version of Bentley-Ottman



- $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$
- $Sw = \{e_0, e_1\}$; compute:

$$det(e_2, e_0) = \begin{pmatrix} m & n & 1 \\ p & q & 1 \\ a & b & 1 \end{pmatrix} > 0 \Rightarrow e_2 \text{ after } e_0 \text{ in } Sw$$

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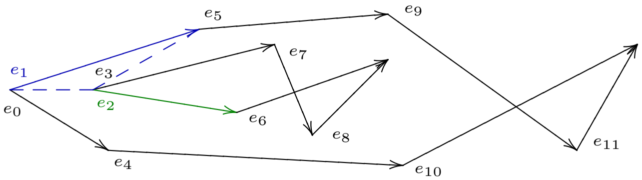
Algorithm 1 - Adapted version of Bentley-Ottman



- $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$
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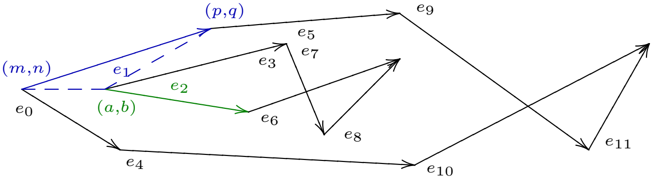
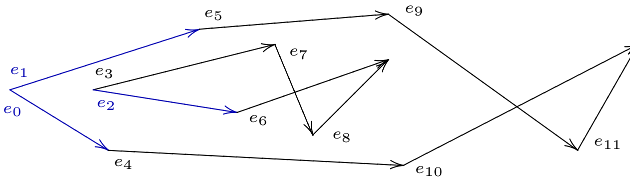
Algorithm 1 - Adapted version of Bentley-Ottman



- $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$
- $Sw = \{e_0, e_1\}$; compute:

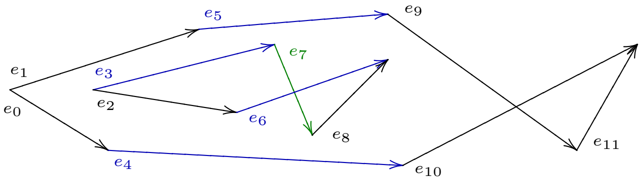
$$det(e_2, e_1) = \begin{pmatrix} m & n & 1 \\ p & q & 1 \\ a & b & 1 \end{pmatrix} < 0 \Rightarrow e_2 \text{ before } e_1 \text{ in } Sw$$

Algorithm 1 - Adapted version of Bentley-Ottman



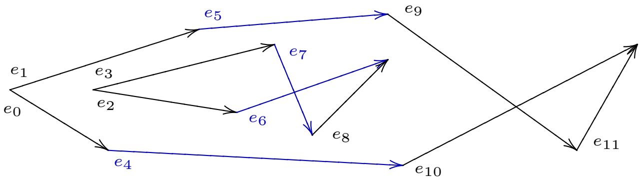
- $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$
- $Sw = \{e_0, e_2, e_1\}$
- Test $e_0 \cap e_2$? No!
  Test $e_2 \cap e_1$? No!
- $I = \emptyset$
  $E_I = \emptyset$

Algorithm 1 - Adapted version of Bentley-Ottman



- $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$
- $Sw = \{e_4, e_6, e_3, e_5\}$

Algorithm 1 - Adapted version of Bentley-Ottman



- $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$
- $Sw = \{e_4, e_6, e_7, e_5\}$
- Test $e_6 \cap e_7 =?$ Yes!
  Test $e_7 \cap e_5 =?$ No! $\Rightarrow I = \{(a_1, b_1)\}\ E_I = \{(e_6, e_7)\}$
  $Sw = \{e_4, e_7, e_6, e_5\}$

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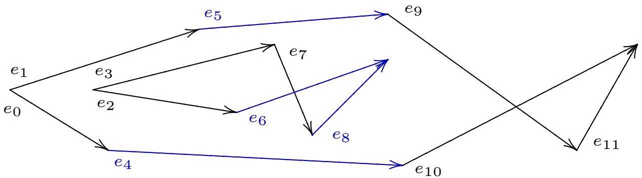
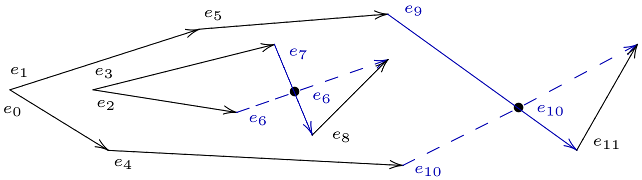
Algorithm 1 - Adapted version of Bentley-Ottman



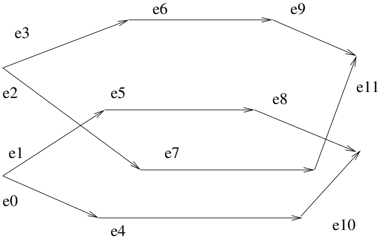
- $E = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$
- $Sw = \{e_4, e_8, e_6, e_5\}$
- Test $e_4 \cap e_8 =$?No!
  Test $e_8 \cap e_6 =$?No!
- Test $dest(e_4) = dest(e_8)$? No!
  Test $dest(e_8) = dest(e_6)$? Yes! $\Rightarrow Sw = \{e_4, \cancel{e_8}, \cancel{e_6}, e_5\} = \{e_4, e_5\}$

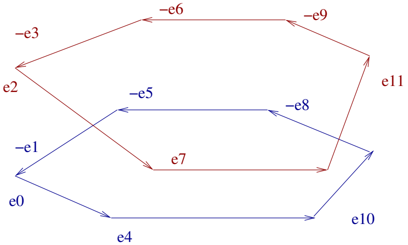
Algorithm 1 - Adapted version of Bentley-Ottman



- The adapted Bentley-Ottman algorithm produces the final output:
  $I = \{i_1 = (x_1, y_1), i_2 = (x_2, y_2)\}$
  $E_I = \{(e_6, e_7), (e_{10}, e_9)\}$ with
  - e_6 below e_7 in $\mathbb{R}^3$ and
  - e_{10} below e_9 in $\mathbb{R}^3$

Algorithm 2 - Constructing the loops

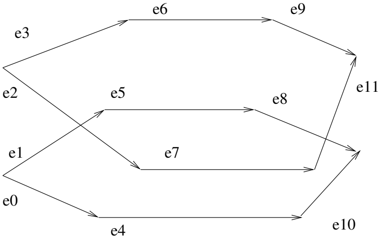


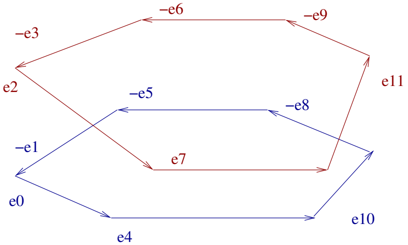


- E ordered by (1),(2),(3) $\Rightarrow$

$$L_0 = \{e_0, e_4, e_{10}, -e_8, -e_5, -e_1\}$$
$$L_1 = \{e_2, e_7, e_{11}, -e_9, -e_6, -e_3\}$$

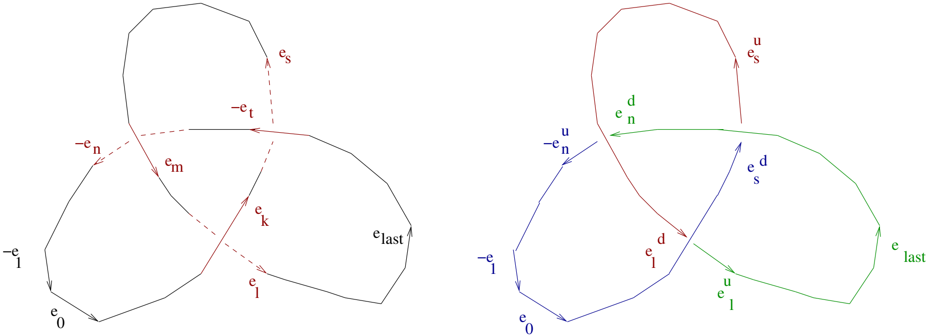
Algorithm 2 - Constructing the loops





- E ordered by (1),(2),(3) $\Rightarrow$ $L_0 = \{e_0, e_4, e_{10}, -e_8, -e_5, -e_1\}$
  $L_1 = \{e_2, e_7, e_{11}, -e_9, -e_6, -e_3\}$
- Here, we introduce:
  ✏ the positive edges ($\xrightarrow{e}$): $x.dest(e) > x.source(e)$
  ✏ the negative edges ($\xleftarrow{-f}$): $x.dest(-f) < x.source(-f)$

Algorithm 3 - Constructing the arcs



- $E = \{e_0, ..., e_{last}\}$
- $E_I = \{(-e_n, e_m), (e_l, e_k), (e_s, -e_t)\}$ $\Rightarrow$
- $L_0 = \{e_0, ..., e_s, e_l, ..., -e_1\}$

$a_0 = \{e_n^u, .., -e_1, e_0, .., e_k, .., e_s^d\}$

$a_1 = \{e_l^u, .., -e_t, .., -e_n^d\}$

$a_2 = \{e_s^u, ..., e_m, ..., e_l^d\}$

- While constructing the arcs we also decide the type of crossings (RH or LH).

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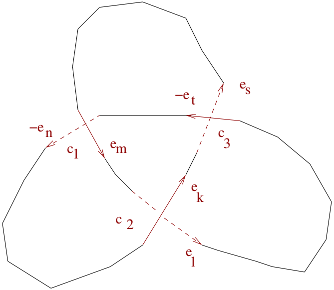
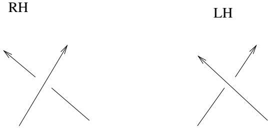
Algorithm 3 - Deciding the type of crossing





- For instance $c_1 = (-e_n, e_m)$ is LH since:
  - x.source$(-e_n)>x$.dest$(-e_n)$,
  - x.source$(e_m)<x$.dest(e_m),
  - slope$(e_m)<$slope$(-e_n)$
- $c_2 = (e_l, e_k)$ is LH, $c_3 = (e_s, -e_t)$ is LH.

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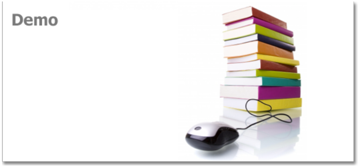
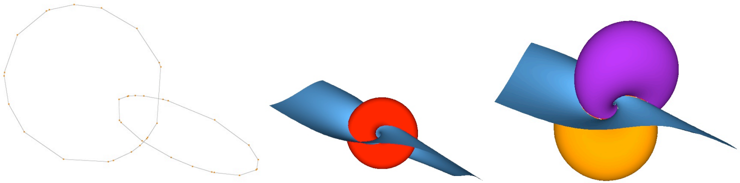
Summary

We have a symbolic-numeric algorithm (i.e. approximate algorithm) for performing operations on a plane complex algebraic curve, implemented in the library GENOM3CK.
About GENOM3CK: `http://people.ricam.oeaw.ac.at/m.hodorog/software.html`

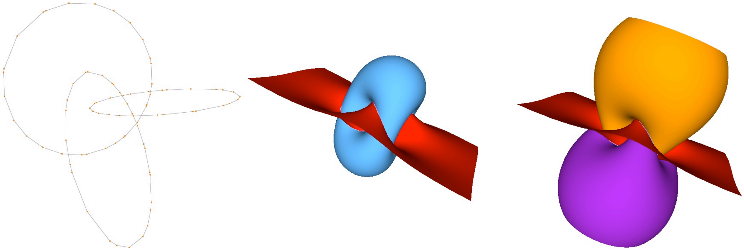
Equation	Link
$x^2 - y^2, \epsilon = 1.0$	Hopf link
$x^2 - y^3, \epsilon = 1.0$	Trefoil knot
$x^3 - y^3, \epsilon = 1.0$	3-knots link
$x^2 - y^4, \epsilon = 1.0$	2-knots link
$x^2 - y^5, \epsilon = 1.0$	1-knot link
$x^4 + x^2y + y^5, \epsilon = 0.5$	3-knots link



Summary (pictures made with GENOM3CK in Axel)



$x^2 - y^2$



$x^3 - y^3$

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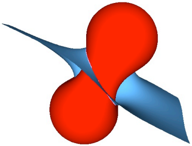
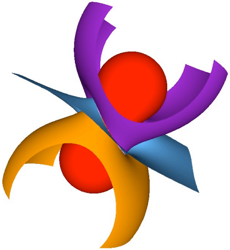
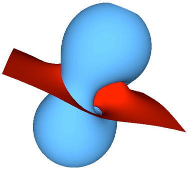
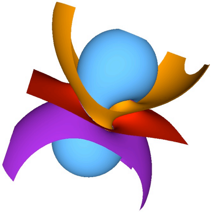
Summary (pictures made with GENOM3CK in Axel)

$x^2 - y^4$

$x^2 - y^5$

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Conclusion and future work

✓ DONE:

- complete automatization of the approximate algorithm (in GENOM3CK); we compute the singularities, topology/algebraic link, Alexander polynomial, delta-invariant, genus;
- experiments show the output is unique and continuously depends on the data;

✗ TO DO's:

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Conclusion and future work

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X TO DO's:

- prove the properties of the approximate algorithm (i.e. convergency, continuity);

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- we can describe it with principles from regularization theory, approximate algebraic computation.

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Conclusion and future work

✓ DONE:

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- experiments show the output is unique and continuously depends on the data;
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X TO DO's:

- prove the properties of the approximate algorithm (i.e. convergency, continuity);
- make precise the meaning of the computed output with the approximate algorithm.

Thank you for your attention.
Questions?