A symbolic-numeric algorithm for genus computation

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Describing the problem

- Input:
  - $\mathbb{C}$ field of complex numbers;
  - $F \in \mathbb{C}[z, w]$ irreducible with coefficients of limited accuracy [1];
  - $C = \{(z, w) \in \mathbb{C}^2 | F(z, w) = 0\} =$
    $= \{(x, y, u, v) \in \mathbb{R}^4 | F(x + iy, u + iv) = 0\}$ complex algebraic curve (d-degree, $Sing(C)$ set of singularities);
- Output:
  - approximate $genus(C)$ s.t.

$$genus(C) = \frac{1}{2}(d-1)(d-2) - \sum_{P \in Sing(C)} \delta\text{-invariant}(P);$$

[1] For now: symbolic coefficients

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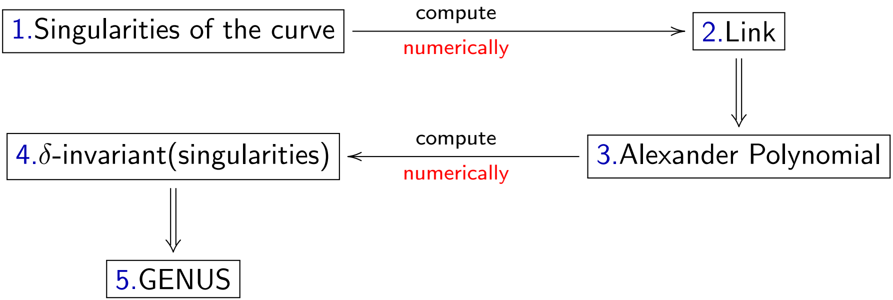
Solving the problem

- Strategy for computing the genus



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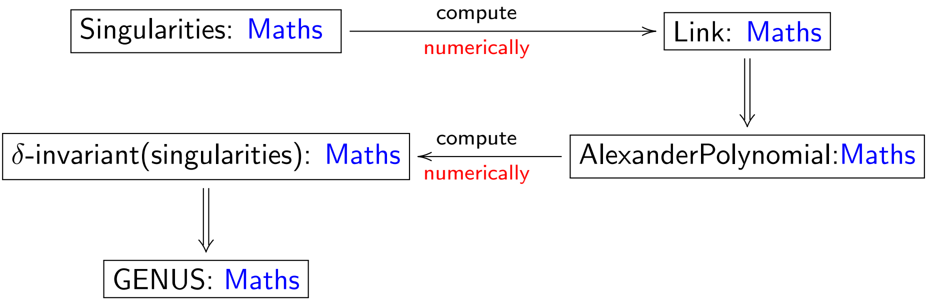
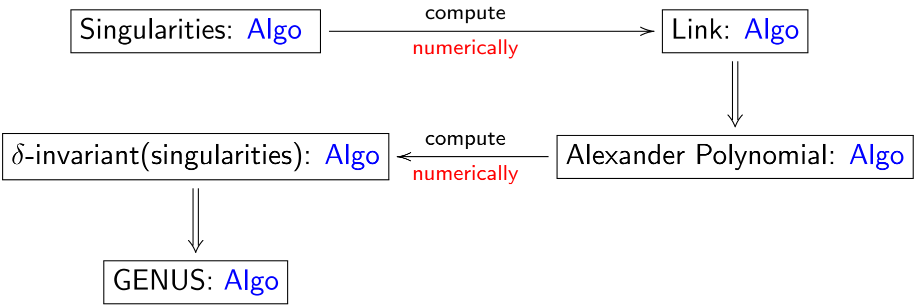
Solving the problem

- Method for computing the genus



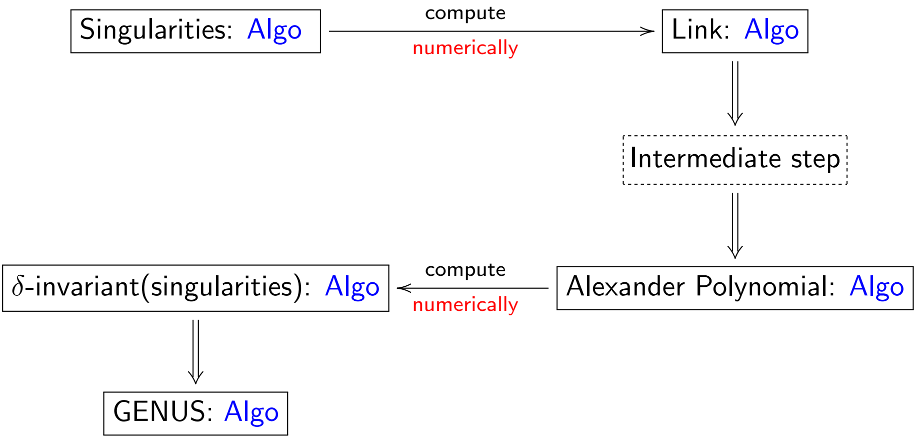
Solving the problem

- Algorithm for the method



Solving the problem

- Algorithm for the method



Solving the problem

Implementation of the algorithm

- *Mathematica* computer algebra system
- *Axel* algebraic geometric modeler
  - developed by *Galaad* team (INRIA Sophia-Antipolis);
  - written in Qt scripting language;
  - topology of implicit curves;
  - intersections of implicit surfaces.

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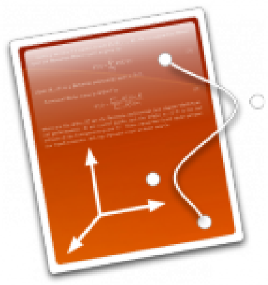

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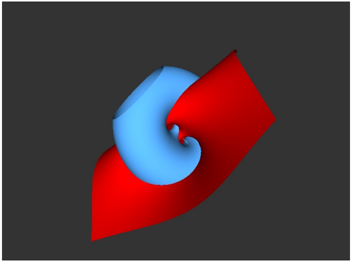

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Computing the link of the singularity

Method (based on Milnor's results)

1. Let $C = \{(x,y,u,v) \in \mathbb{R}^4 | F(x+iy, u+iv) = 0\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$, with $\left(F(0,0), \frac{\delta F}{\delta z}(0,0), \frac{\delta F}{\delta w}(0,0)\right) = (0,0,0)$, where $z = x+iy, w = u+iv.$

2. Consider $S^3 = \{(x,y,u,v) \in \mathbb{R}^4 | x^2+y^2+u^2+w^2 = \epsilon^2\} \subset \mathbb{R}^4$ and $X = C \bigcap S^3 = \{(x,y,u,v) \in \mathbb{R}^4 | F(x,y,u,v) = 0, x^2+y^2+u^2+w^2 = \epsilon^2\}.$

3. For $P(0,0,0,\epsilon) \in S^3 \setminus C$, construct
   $f : S^3 \setminus \{P\} \subset \mathbb{R}^4 \to \mathbb{R}^3, (x,y,u,v) \to (a,b,c) = (\frac{x}{\epsilon - v}, \frac{y}{\epsilon - v}, \frac{u}{\epsilon - v})$
   $f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) \in C \bigcap S^3 : (a,b,c) = f(x,y,u,v)\}$
   $f(X)$ is a link.

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Computing the link of the singularity

Method (next)

3. $$f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) \in C \bigcap S^3 : (a,b,c) = f(x,y,u,v)\}$$
$$f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) = f^{-1}(a,b,c) \in C \bigcap S^3\}$$

4. Compute $f^{-1} : \mathbb{R}^3 \to S^3 \setminus \{P\}$
$$(a,b,c) \to (x,y,u,v) = \left(\frac{2a\epsilon}{1+a^2+b^2+c^2}, \frac{2b\epsilon}{1+a^2+b^2+c^2}, \frac{2c\epsilon}{1+a^2+b^2+c^2}, \frac{\epsilon(a^2+b^2+c^2-1)}{1+a^2+b^2+c^2}\right)$$

5. Get $$f(X) = \Big\{(a,b,c) \in \mathbb{R}^3 | \\ F\left(\frac{2a\epsilon}{1+a^2+b^2+c^2}, \frac{2b\epsilon}{1+a^2+b^2+c^2}, \frac{2c\epsilon}{1+a^2+b^2+c^2}, \frac{\epsilon(a^2+b^2+c^2-1)}{1+a^2+b^2+c^2}\right) = 0\Big\} \Leftrightarrow$$
$$f(X) = \{(a,b,c) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$$

Computing the link of the singularity

Method (next)

3. $$f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) \in C \bigcap S^3 : (a,b,c) = f(x,y,u,v)\}$$
$$f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) = f^{-1}(a,b,c) \in C \bigcap S^3\}$$

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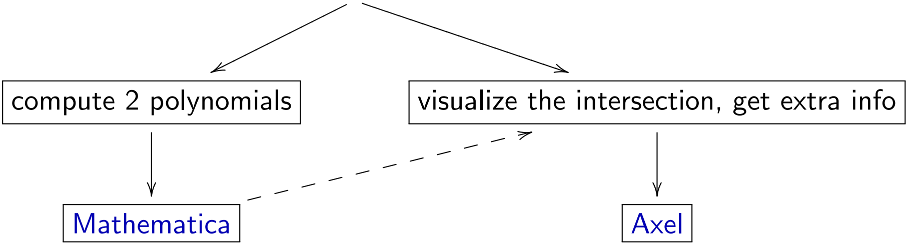
Compute B s.t.
$$f(X) = \{(a,b,c) \in B \subset \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$$ is a link

Computing the link of the singularity

Method (summary)

$$f(X) = \{(a,b,c) \in B \subset \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$$



Computing the link of the singularity

Why Axel?

Axel computes the topology of implicit curves in $\mathbb{R}^3$.

- Input:
  - $f, g \in \mathbb{R}[x, y, z]$
  - $C = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = 0, g(x, y, z) = 0\}$
  - $D = [a_0, b_0] \times [a_1, b_1] \times [a_2, b_2], \epsilon \geq 0$
- Output:
  - $Graph(C) =< \mathcal{V}, \mathcal{E} >$ with
    $\mathcal{V} = \{p = (a, b, c) \in \mathbb{R}^3\}$
    $\mathcal{E} = \{(i, j) | i, j \in \mathcal{V}\}$
- s.t. $Graph(C) \cong_{isotopic} C$

Computing the link of the singularity

Algorithm

- Get the 2 polynomials
- Compute the box B
- Generate the Axel file

Implementation-MMa

- formEqns$[z^2 - w^2, 1]$
- getBoxValue$[z^2 - w^2, 1]$
- genAxelFile$[z^2 - w^2, 1, "ex.axl"]$

- Note: We run the obtained file "ex.axl" with Axel.

Computing the link of the singularity

Test experiments (with Axel)

Equation	Tests on ϵ					
	ϵ=0.5		ϵ=1.0		ϵ=4.3	
	$[-b,b]^3$	link	$[-b,b]^3$	link	$[-b,b]^3$	link
z^2-w^2	2.41421	**Hopf link**	2.41421	Hopf link	2.41421	Hopf link
z^2-w^3	3.38298	**Trefoil knot**	2.67567	Trefoil knot	1.84639	Trefoil knot
$z^2-w^2-w^3$	2.37636	**Hopf link**	2.28464	Curve one singularity	2.24247	Trefoil knot

V.I. Arnold's results: $Top(z^2-w^2-w^3) \cong Top(z^2-w^2)$

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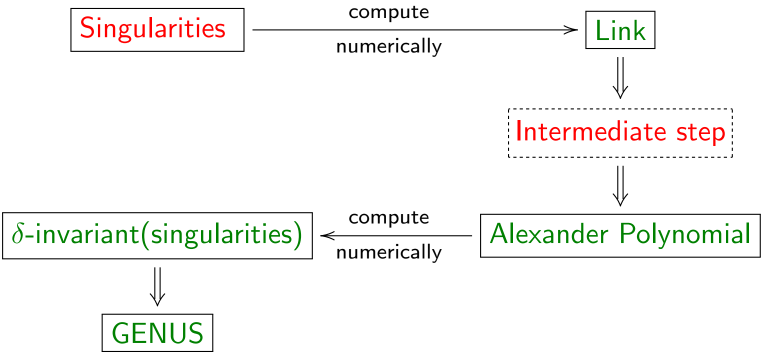
Summary

- At present: for symbolic coefficients



- Future work: tests for algorithm with numeric coefficients

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Conclusion

- first results and test experiments were presented;
- Future work:
  - deeper introspection into some mathematical aspects (i.e. Milnor's fibration, Alexander polynomial);
  - correctness/completeness for the algorithm;
  - implementation of the algorithm;
  - analysis of the algorithm.

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Thank you for your attention.