Singularities and knots

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Motivation

- DK Project: Symbolic-Numeric Techniques for Genus Computation and Parametrization

Symbolic Computation

- Pros
  - exact algorithms exist for a large class of problems;
- Cons
  - expensive (time, memory);
  - no analytic solution for some problems;

Numeric Computation

- Pros
  - cheap (time, memory)
  - always has a numerical solution to the problem
- Cons
  - need control of the numerical errors

- $\Rightarrow$ Symbolic-Numeric computation.

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Preliminaries

- Algebraic Geometry

Basic Notions
plane algebraic curve
singularity of a curve
genus of a curve

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Definition. Let K be an algebraically closed field, and $f(x, y) \in K[x, y]$ a nonconstant squarefree polynomial. A plane algebraic curve over K is defined as the set $C = \{(x, y) \in K^2 | f(x, y) = 0\}$; f is called the defining polynomial of C.

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Definition. Let $C = \{(x,y) \in K^2 | f(x,y) = 0\}$ be a plane algebraic curve, and $(a,b) \in C$ (i.e. $f(a,b) = 0$). The point (a,b) is a singularity of C iff

$$\left(\frac{\delta f}{\delta x}(a,b), \frac{\delta f}{\delta y}(a,b)\right) = (0,0).$$

Preliminaries

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Definition. Let C be a plane algebraic curve, $Sing(C)$ the set of singularities of C, and d the degree of C. Then:

$$genus(C) = \frac{1}{2}(d-1)(d-2) - \sum_{P \in Sing(C)} \delta\text{-invariant}(P).$$

Theorem. A plane algebraic curve C is parametrizable iff genus(C)=0.

What?

- In:
  - K a field;
  - $F \in K[x, y]$ irreducible with coefficients of limited accuracy;
    - coefficients as exact data $(\alpha \in \mathbb{R}, \mathbb{C})$;
    - coefficients as numerical data ($(\bar{\alpha} \in \mathbb{R}, \mathbb{C}, \epsilon \in \mathbb{R}_+)$);
  - $C = \{(x, y) \in K^2 | F(x, y) = 0\}$ plane algebraic curve (d-degree, $Sing(C)$ set of singularities);
- Out:
  - approximate $genus(C)$ s.t.

$$genus(C) = \frac{1}{2}(d-1)(d-2) - \sum_{P \in Sing(C)} \delta\text{-invariant}(P);$$

  - approximate $rationalParametrization(C)$ (if applicable);

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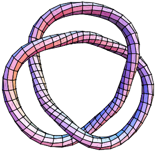
Preliminaries

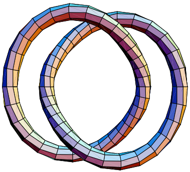
Trefoil Knot



A **knot** is a simple closed curve in $\mathbb{R}^3$.

A **link** is a finite union of disjoint knots.
A knot is a link with one component.

Hopf Link

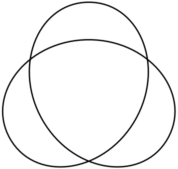


Preliminaries

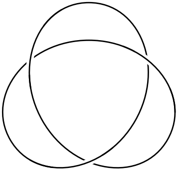
A knot projection is a **regular projection** if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.

A **knot/link diagram** is the image under regular projection, together with the information on each crossings telling which branch goes over and which under.

Regular projection



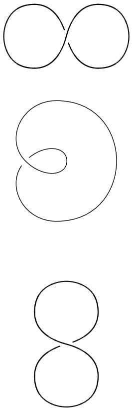
Knot Diagram



Preliminaries

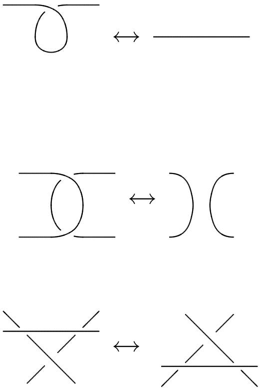
Projections of unknot

Given 2 knots, can we tell whether they are alike or not?



Preliminaries

Reidemeister moves



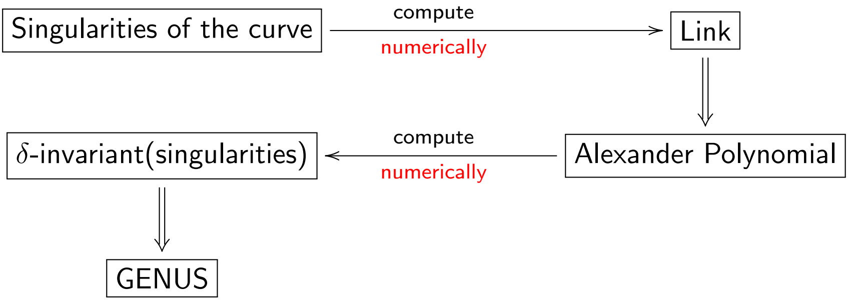
A **knot invariant** is a function from knot diagrams to some discrete set which is invariant under the Reidemeister moves.

Knot invariant: Alexander polynomial.

Two knots/links are equivalent iff some diagram of one can be transformed to some diagram of the other by a finite number of Reidemeister moves.

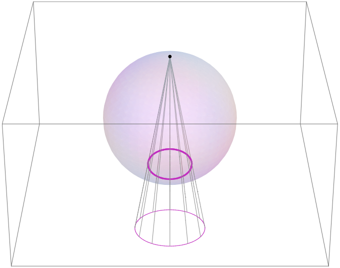
How?

- How do we compute the genus?



How?

- Why the link of a singularity?
  - drawing a singularity over $\mathbb{R}$ is easy.
  - drawing a singularity over $\mathbb{C}$ is not so easy!
    So we look at the link of the singularity.
- How do we compute the link?
  - use stereographic projection;



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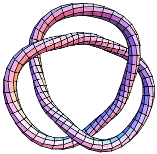
How

How do we compute the link?

- Consider $C \subset \mathbb{C}^2 \cong \mathbb{R}^4$ s.t. $(0,0)$ singularity of C.
- Step 1: Consider $S_{(0,\epsilon)} = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = \epsilon\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$
  - choose a good radius ϵ!
- Step 2: $X = C \cap S_{(0,\epsilon)}$
  - For small ϵ, X is a disjoint union of closed loops.
- Step 3: $P \notin X$, apply stereographic projection $f : (S_{(0,\epsilon)} - P) \to \mathbb{R}^3$
  - $Y = f(X)$ is a link;
- Example: The link of the singularity of the curve $y^2 - x^3 = 0$ is the trefoil knot.



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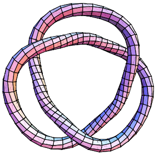

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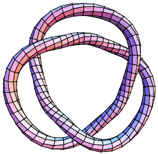

Conclusion

Future Work

- construct the algorithm for the described method;
- realize the implementation of the algorithm;

Thank you for your attention.