Basic Knot Theory

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Motivation

My current research refers to the subject of my PhD Thesis, focused on *"Symbolic numeric algorithms for genus computation of plane complex algebraic curves based on knot theory"*. Among other important mathematical tools (i.e. computational geometry, numerical polynomial algebra), we use combinatorial and algebraic methods from knot theory to design some of our algorithms which turn out to be efficient. During one of my talks, I was draw the attention on the importance and the potential of other methods from knot theory. I will thus try:

- to give a basic summary of the most important aspects of knot theory;
- to show how knot theory can be analyzed from other important significant domains of mathematics;
- **Please:** any person who is interested in one specific subject and wants to study it in greater details, it is more than welcomed and encouraged to do so.

The talk will be more as a "lecture" than as a "presentation".

Introduction

Knot theory, which dates back to the late 19th century, gained increased attention in the last two decades of the 20th century together with its potential applications in physics, chemistry, biology or mathematics. Most of the problems in knot theory are open problems.

But what is knot theory?

Most of the answers one gets or most of the definitions one finds:

"Knot theory is a branch of algebraic topology that is involved in the study of three-dimensional manifolds,i.e. the study of the ways in which knotted copies of a circle can be embedded in three-dimensional space..."

Introduction

What does all of this mean?

Preliminaries

Note: this section requires some notions from topology (i.e. topological space, homeomorphism, homotopy, isotopy, ambient isotopy). We first define these notions.

Definition

Let X, Y topological spaces. $X \sim_{\textit{homeomorphic}} Y$ iff $\exists\ f : X \to Y$ s.t. f is continuous and bijective and f^{-1} is continuos (f is bicontinuos).

Definition

Let X, Y topological spaces, $f, g : X \to Y$ continuous functions.
$f \sim_{\textit{homotopic}} g$ iff $\exists\ G : X \times [0,1] \to Y$ continuous s.t.
$\forall x \in X : G(x,0) = f(x), G(x,1) = g(x)$. G is called a homotopy.

Preliminaries

Definition

Let X, Y *topological spaces.* $f, g : X \to Y$ *homeomorphisms.* $f \sim_{\text{isotopic}} g$ *iff* $\exists\ H : X \times [0,1] \to Y$ *homotopy s.t.* $H_t = H|X \times t$ *is an homeomorphism for* $0 \leq t \leq 1$. *H is called an isotopy.*

Definition

Let $X, Y \subset \mathbb{R}^3$ *topological spaces.* $X \sim_{\text{ambientIsotopy}} Y$ *iff* $\exists\ F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ *continuous s.t. all* $F_t = F|F \times t$, *with* $0 \leq t \leq 1$ *are homeomorphisms from* $\mathbb{R}^3$ *to* $\mathbb{R}^3$ *and* $F(\cdot, \circ)$ *is the identity on* Y *and* $F(X, 1) = Y$.

Definition and equivalence of knots

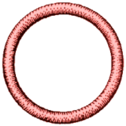
Definition

A knot $K \subset \mathbb{R}^3$ or S^3 is a subset of points homeomorphic to the circle (S^1).

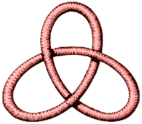
Remark: The circle can be arranged as a smooth curve or polygonal curve.

Examples of knots:

- unknot (trivial knot)



- trefoil knot



Definition and equivalence of knots

How would the idea of equivalence be captured?
(i.e. deformation of a knot K in $\mathbb{R}^3$?)

- Let $h : K \times [0,1] \to \mathbb{R}^3$ a homotopy. If $t \in [0,1]$ indicates time, then $h_t(K)$ for increasing values of t show the evolution of K in $\mathbb{R}^3$ (h_0 is the identity map, all h_t are continuous). Homotopy allows the knot to intersect itself, so it is not useful! Why?

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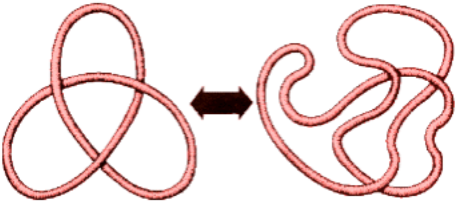
- Let all h_t injective (i.e. h is an isotopy). Isotopy allows to shrink a part of the knot down to a point, so it is not useful! Why?
  - $\Rightarrow$ All knots are isotopic to the trivial knot!
- We need to consider h an ambient isotopy, which allows to deform the knot through the space $\mathbb{R}^3$ in which it sits in. (i.e. an ambient isotopy of $K \subset \mathbb{R}^3$is an isotopy that carries K with it).

Definition and equivalence of knots

Definition

Two knots $K_1, K_2 \subset \mathbb{R}^3$ *are ambient isotopic (isotopic) iff* $\exists\ h : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ *an isotopy s.t.* $h(K_1, 0) = h_0(K_1) = K_1$ *and* $h(K_1, 1) = h_1(K_1) = K_2$. *We denote* $K_1 \sim_{ambientIsotopic} K_2$ *or* $K_1 \sim_{isotopic} K_2$.

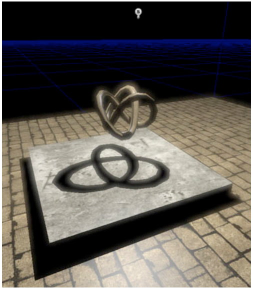
Example of knot equivalence:



Definition and equivalence of knots

Remarks:

- When we work with knots we work with their projections in $\mathbb{R}^2$ with additional information. Such projections are much easier to work with, but they are "artificial": knot theory is concerned mainly with 3-dimensional topology.



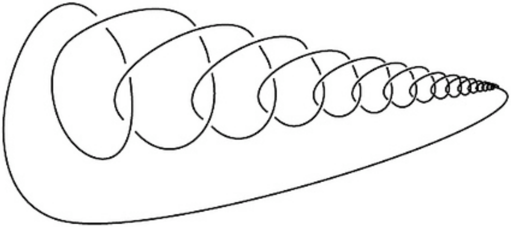
- So far we have defined knots in general. We can specify a particular knot using the following definition.

Definition and equivalence of knots

Definition

A knot is a continuous simple closed curve in $\mathbb{R}^3$.

Remark: The continuity in this definition allows the existence of infinitely knotted loops (wild knots). To eliminate the wild points in the wild knots (the point where the small knots bunch up) we can introduce differentiability. This eliminates the wild points, since there is no continuous way to define a tangent direction at this point. This remedy is possible but difficult.



Definition and equivalence of knots

Alternatively, we can use polygonal curves to define knots (they are finite by nature). We can think of knots as build up of straight lines and give the coordinates of the corners.

Definition

Let $p, q \in \mathbb{R}^3, p \neq q$. *Let* $[p, q]$ *denote the segment line joining* p, q. *Let* $(p_1, ..., p_n)$ *be an ordered set with* $p_i \neq q_j, i, j \in \{1, 2, ..., n\}$. *Then* $P = \bigcup_{i=1}^{n-1} [p_i, p_{i+1}] \cup [p_n, p_1]$ *is a closed polygonal curve. P is simple iff each segment intersects exactly two other segments only at their endpoints.*

Definition

A knot K *is a simple closed polygonal curve* P *in* $\mathbb{R}^3$. *The line segments of* P *are called the edges of the knots, and the corners of* P *are called the vertices of the knot. A knot is called tame if it has a polygonal representative.*

Definition and equivalence of links

Definition

A link is a finite disjoint union of knots $L = K_0 \cup K_1 \cup ... \cup K_n$*. Each knot* K_i *is called a component of the link. The number of components of a link is called the multiplicity of the link* $\mu(L)$*. A subset of the components of* L *embedded in the same way is called a sublink.*

Remarks:

- A knot is a link with one component
- The unlink (trivial link) is the union of unknots all lying in the same plane
- Links can be oriented (each component is assigned an orientation). An unoriented n-component link can be assigned orientations in...

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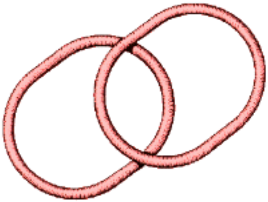
- ...2^n ways.

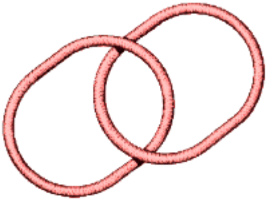
Definition and equivalence of links

Examples of links:

- unlink (trivial link)



- Hopf link



Definition and equivalence of links

Definition

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Remarks:

- This isotopy has to preserve any orientations or labeling on the links. Without this requirement, the definition is weak as it does not impose any restriction on the isotopy: there is a free choice of how to match the components of L_1 with those of L_2.

Projections and diagrams of knots

Definition

Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map with $(x, y, z) \to (x, y)$. If K is a knot then the image of K under p is called the projection of K (i.e. $K \to p(K)$).

Definition

A knot projection is regular iff no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot. A crossing point is the image of two knot points of such a regular projection from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Projections and diagrams of knots

Remark: Regular projections are useful due two 2 theorems according to which (see C. Livingston, Knot theory for complete proofs):

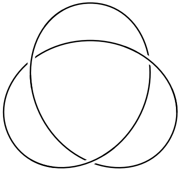
- if a knot does not have a regular projection then there exists an equivalent knot nearby that does have a regular projection.
- if a knot does have a regular projection then all nearby knots are equivalent and have regular projections.

Definition

A diagram is the image under regular projection together with the information on each crossing telling which branch goes under and which goes over. We thus speak about overcrossings and undercrossings. An arc is the part of the diagram between two undercrossings.

Projections and diagrams of knots

Example of diagram:



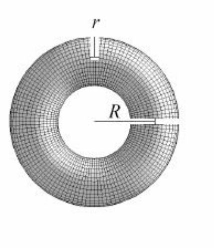
Remarks:

- The crossing points correspond to double points in the projection.
- The number of arcs equals the number of crossings.

Summary:

- Knot is a subset of $\mathbb{R}^3$. Knots determine equivalence classes of knots.
- Knots with regular projections have diagrams (which are drawings in the plane).

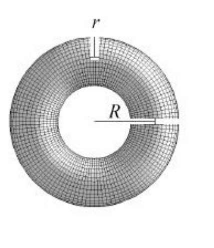
Types of knots: Torus knots



A torus is generated by taking a circle in the yz plane of radius r centred on the y-axis at distance $R+r$ from the origin, then rotating it around the z-axis. If we parametrize the circle by angle $\theta \in [0, 2\pi]$ and the rotation by angle $\phi \in [0, 2\pi]$ we can express the torus as:

$$\begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ R + rcos\theta \\ rsin\theta \end{pmatrix} = \begin{pmatrix} -sin\phi R + rcos\theta \\ cos\phi R + rcos\theta \\ rsin\theta \end{pmatrix}$$

Types of knots: Torus knots

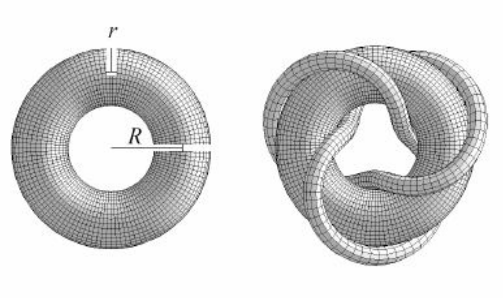


The parameters r, R control the geometry of the torus: r is the radius of the tube, R is the radius of the hole. The angles form a coordinate system: any point on the torus can be labelled by a pair (θ, ϕ).

The subset of points defined by the equation $p\theta = q\phi$ for coprime integers p, q winds its way around the torus and forms a knot, called a (p, q)-torus knot. A (p, q)-torus knot is equivalent with a (q, p)-torus knot. Since they lie in the surface of a standard torus, torus knots are some of the simplest knots to describe parametrically.

Types of knots: Torus knots

Example: Trefoil knot is a (3,2)-torus knot



Definition

1. A (p, q)-torus knot is obtained by looping a string through the hole of a torus p times with q revolutions before joining its ends, where p and q are relatively prime.
2. A (p, q)-torus knot is a curve on the torus, which is specified by winding p times around the main axis of the torus and q times around the tube of the torus.

Types of knots: Prime knots

What is the sum of two knots?
Intuition: Two oriented knots (or links) can be summed by placing them side by side and joining them by straight bars so that orientation is preserved in the sum.
Remarks:

- The knot sum is also known as composition (Adams 1994) or connected sum (Rolfsen 1976)
- Given any 2 knots K, J one can form their connected sum denoted $K\#J$.
- Example: Connected sum of the trefoil and the figure eight knot.

Types of knots: Prime knots

Definition

A knot is prime if it cannot be decomposed as a connected sum of nontrivial knots.

Remarks:

- All torus knots are prime knots (Hoste et al. 1998, Burde and Zieschang 2002). Figure eight knot is a prime knot.
- Schubert proved that any knot can be decomposed uniquely as the connected sum of prime knots (1974). (he used geometric methods). This is an analogy to positive integers.
- $K(trefoil\#unknot) = trefoil.$

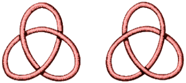
Types of knots: Amphicheiral knots

Definition

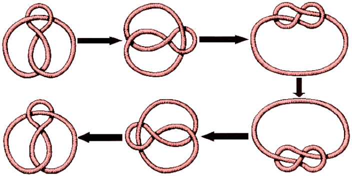
A knot that is equivalent to its mirror image is called amphicheiral (or achiral).

Examples:

- Trefoil not is not amphicheiral (Jones polynomial distinguishes the trefoil knot from its mirror image).



- Figure eight knot is amphicheiral .



Types of knots: Alternating knots

Definition

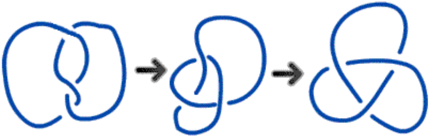
An alternating knot is a knot which possesses a diagram in which crossings alternate between under and over crossings. Not all knot diagrams of alternating knots need to be alternating diagrams.

Remarks:

- Trefoil, figure eight knots are alternating knots;
- All prime knots with 7 or fewer crossings are alternating knots;

Reidemeister moves and theorem

How can we tell if two diagrams represent the same knot?



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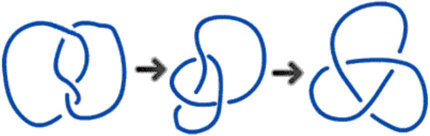


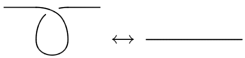
- This is the ultimate question in knot theory and there is no algorithm for finding an answer to it!
- But Kurt Reidemeister made a good start on the problem by proving that 2 diagrams representing the same knot are always related by a sequence of 3 special moves called the Reidemeister moves (Reidemeister theorem-1926).
- Having this theorem makes it much easier to find invariants on knots based on diagrams. To check that something is an invariant, we would only need to show that it does not change under any single Reidemeister move.

Reidemeister moves and theorem

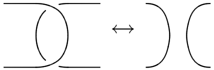
Definition

A Reidemeister move is one of the 3 ways to change a diagram of a knot that will change the relation between its crossings.

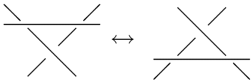
- First Redemeister move:



- Second Redemeister move:



- Third Redemeister move:



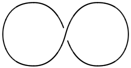
Reidemeister moves and theorem

Theorem (Reidemeister)

Two links are equivalent iff any diagram of one can be transformed into a diagram of the other by a sequence of Reidemeister moves.

Remarks:

- Projection of the unknot:



- If two links are equivalent it can be difficult to decide the smallest number of moves that can be used to transform one diagram into the other. If after some time one cannot transform one diagram into another, maybe one is not smart enough! Or it might just happen that the knots are really different.
- In order to show that 2 knots are different we use knot/link invariants!

Invariants of knots/links

Definition

A link invariant is a function from the set of links to some other set whose value depends only on the equivalence class of the link. Any representative from the class can be chosen to calculate the invariant. There is no restriction on the kind of objects in the target space (i.e. integers, polynomials, matrices or groups).

Some invariants of knots/links:

- numeric invariants: unknotting number, crossing number.
- colorability (and generalization of colorability: mod p labellings).
- polynomial invariants (Alexander, Jones, HOMFLY polynomials).

Invariants of knots: Crossing number, Unknotting number

Basic facts:

- The crossing number and the unknotting number are the most natural invariants to study knots.
- They are simple to define, but their computation is difficult and the most natural questions about them are unanswered.
- The unknotting number remains unknown for many knots!

Invariants of knots: Crossing number, Unknotting number

Definition

1. The crossing number of a knot K, denoted with $c(K)$ is the least number of crossings that occur in any diagram of the knot.
2. A knot K has unknotting number n (denoted $u(K) = n$) if $\exists$ a diagram of K s.t. changing n crossings in the diagram turns the knot into the unknot and there is no other diagram s.t. fewer changes would have turned it into the unknot.

- c(trefoil)=3, u(trefoil)=1.

- c(cinquefoil)=5, u(cinquefoil)=2.

Invariants of knots: Crossing number, Unknotting number

Open problems

- Behaviour of $c(k)$ under connected sum!?

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Invariants of knots: Crossing number, Unknotting number

Open problems

- Behaviour of $c(k)$ under connected sum!?
  - Presently, it is conjectured but unproven, that $c(K)$ adds under connected sum (this was proven for alternating knots only).
  - Still we can not rule out the possibility that the connected sum of two knots can have crossing number less than either factor!
- Behaviour of $u(k)$ under connected sum!?
  - Weaker results: the connected sum of two knots with unknotting number 1 is a knot with unknotting number 2 (Scharlemann).

Invariants of knots: Colorability

Definition

A diagram of a knot is colorable iff each arc can be drawn using one of the 3 colors s.t. at each crossing either 3 different colors came together or the same color comes together and at least 2 of the colors are used.

- Trefoil is colorable.

- Unknot is not colorable. Why?

- Conclusion: the trefoil and the unknot are different knots. Any colorable knot is nontrivial.

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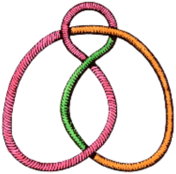
- Unknot is not colorable. Why?
  - We use only one color to draw the unknot (but we need at least 2).
- Conclusion: the trefoil and the unknot are different knots. Any colorable knot is nontrivial.

Invariants of knots: Colorability

Remarks:

- Figure eight knot is not colorable! Why?



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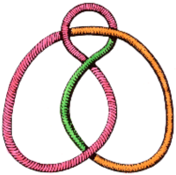

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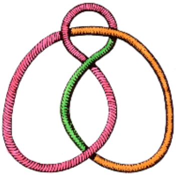

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- Colorability is not a complete invariant for knots! Why?
  - Figure eight and unknot are not colorable, so colorability cannot be used to show that figure eight is different from the unknot (and thus nontrivial).

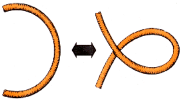
Invariants of knots: Colorability

Theorem

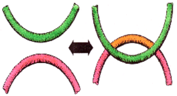
If a diagram of a knot is colorable then all of its diagrams are colorable.

Proof: If a Reidemeister move is performed on a colorable diagram then the resulting diagram is colorable.

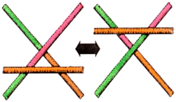
1. We stay with one color on both sides.

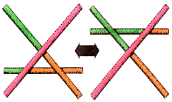


2. On the right we have three colours, and on the left no crossing.



3. At a move some colors can be changed, but fulfilling the basic rule: at each crossing, there must be either 1 colour or 3 colours.





Invariants of knots: Generalization of colorability

What about other types of colouring?

- It is interesting to experiment with more than 3 colours, in the same way, but it has been found that this does not directly give a knot invariant. Instead one has to use a more subtle kind of labelling, which does give new invariants, i.e. mod p labellings, where p is a prime positive integer.
- That is why one wants to find the number of mod p labelings of a knot diagram.

Invariants of knots: Generalization of colorability

How many mod p labelings of a knot diagram are there?

- Finding mod p labelings of a knot diagram can be reduced to solving a system of linear equations mod p.
- The dimension of the solution space for this system of equations is called the mod p rank of the knot.
- It is proved that if K has mod p rank n, then the number of the mod p labelings of the knot diagram is $p(p^n - 1)$ (see C. Livingston, Knot theory for details) .

Invariants of knots: Polynomial invariants

Introduction:The polynomial invariants are the most successful ways to tell knots apart:

- *Alexander Polynomial* (1928): it distinguishes all knots of 8 crossings or fewer, but it does not distinguish a knot from its mirror image (i.e. amphicheiral knots).
- *Jones polynomial* (1984): it distinguishes all knots of 10 crossings or fewer, a knot from its mirror image, but it does not distinguish mutant knots.
- *HOMFLY polynomial* (1985/87): it is a generalization of both Alexander and Jones polynomials. It is named after its inventors Hoste, Ocneanu, Millet, Freyd, Lickorish, Yetter (independently also Prztycki and Traczyk discovered the same polynomial). It does not distinguish mutant knots.

Presently, there is no complete polynomial invariant for knots!

Invariants of knots: Alexander polynomial

Introduction:Alexander polynomial can be computed in several ways:

- *Alexander's combinatorially method* (1928): it uses the diagram of the knots, and the Reidemeister moves (this method was presented at least 4 times as part of the work concerning the DK9 project).
- *Fox's method* (1963): it uses a representation of the fundamental group of the complement of the knot. It was also mentioned in Alexander's original paper in his "Miscellaneous" section, but Fox's description is more detailed.
- *Conway's skein relation* (1969): it uses skein relation, some special equations that connect the crossings of different knot diagrams. It was also mentioned in Alexander's original paper, but Conway's presentation is clearer and thus it paved the discovery for the Jones polynomial 15 years later.

Invariants of knots: Alexander polynomial (Fox's method)

Preliminaries

- Fox's method is based on the study of the properties of the fundamental groups of the complementary spaces of knots.
- the complementary space of a knot K denoted with $\mathbb{R}^3 \setminus K$ consists of all the points of $\mathbb{R}^3$ that do not belong to K.
- We remember that $K \subset \mathbb{R}^3$ is a knot iff there exists a homeomorphism of the unit circle S^1 into $\mathbb{R}^3$ whose image is K.
- Problem: *How do we define the fundamental group of the knot complement?*

The fundamental group of a topological space

Definition

A path in the topological space X is a continuous map $p : [0,1] \to X$.
The continuos function $f : [0,1] \to X$ with the property that $f(0) = f(1)$ is a loop.
$f(0) = f(1)$ is called the base point of the loop.

Definition

Let $x \in X$, and denote with $P(X, x)$ the set of all loops in X with the base point x.
Let $f, g \in P(X, x)$. Two loops f, g are homotopic ($f \sim_{\text{homotopic}} g$) iff
$\exists\ H : [0,1] \times [0,1] \to X$ continuous (i.e. the family of maps
$H_t = H|[0,1] \times \{t\}, t \in [0,1]$ is also continuous) s.t.:
1. $H_0(y) := H(y,0) = f(y)$ and $H_1(y) := H(y,1) = g(y)\ \forall y \in [0,1]$.
2. The base point $H_t(0) := H(0,t) = H(1,t) := H_t(1) = x$ is independent of t.

Remark: The homotopy relation from this definition is an equivalent relation.

Invariants of knots: Alexander polynomial (Fox's method)

The fundamental group of a topological space (next)

- Notation: We denote with $\pi(X,x) = P(X,x)/\sim_{\textit{homotopy}}$ the collection of all equivalence classes of loops in the topological space X with the base point x.
- Remark: If on $\pi(X,x)$ we define the following operations:
  - Inverse: $i : \pi(X,x) \to \pi(X,x)$
    $s :\in \pi(X,x) \to i(s)(t) := s^{-1}(t) = s(1-t) \in \pi(X,x)$.
  - Multiplication: $m : \pi(X,x)^2 \to \pi(X,x)$
    $(s_1, s_2) \in \pi(X,x)^2 \to m(s_1,s_2) := s_1 \circ s_2 = s_3 \in \pi(X,x)$, where

    $$s_3(t) = \begin{cases} s_1(2\cdot t), t \in [0, \frac{1}{2}] \\ s_2(2\cdot t - 1), t \in [\frac{1}{2}, 1] \end{cases}$$

  then $(\pi(X,x), m, i) := (\pi(X,x), \circ, \ ^{-1})$ is a group called the fundamental group of the topological space X relative to the base point x.

The fundamental group of a connected topological space

Definition

A topological space X is connected iff any two of its points can be joined by a path lying in X.

Remark: $\mathbb{R}^3$ is a connected topological space.

Theorem

If X is a connected topological space with $x, y \in X$ then $\pi(X, x) \sim_{\text{isomorphic}} \pi(X, y)$ (i.e. in a connected topological space X the fundamental group of X is independent on the base point).

Invariants of knots: Alexander polynomial (Fox's method)

The knot group

- The fundamental group of a knot $K \subset \mathbb{R}^3$ for some choice of basepoint $x \in K$ denoted with $\pi(K, x)$ is isomorphic to $\mathbb{Z}$. This group is not so interesting to study due to Dehn's lemma, proved by Papakyriakopoulos (1957).
- The fundamental group of the complement of the knot K denoted with $\pi(\mathbb{R}^3 \setminus K, x)$ is more interesting to study. Since $\mathbb{R}^3 \setminus K$ is a connected topological space, we use simply the notation $\pi(\mathbb{R}^3 \setminus K)$ for the fundamental group of the complement of the knot because $\pi(\mathbb{R}^3 \setminus K)$ is independent of the base point.
- The fundamental group of the complement of the knot K denoted with $\pi(\mathbb{R}^3 \setminus K) := G(K)$ is simply called the knot group of K.
  - Dehn's lemma: If a knot group is isomorphic to the group of integers then the knot is trivial (unknot).

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of a group

- What is the free group?

  Let $S = |n|$ be a set of cardinality n. Then any $a \in S$ is called a letter of S, any a^n with $a \in S, n \in \mathbb{Z}$ is called a syllable, and a finite ordered sequence of syllables such as $b^{-3}a^0a^1c^2c^2a^0c^1$ is called a word. The unique word 1 is called the empty word. We denote with $W(S)$ the set of all words formed on S. On $W(S)$ we define the operations:

  - the product of 2 words formed by writing one word after the other;
  - the elementary expansions and contractions: If $w_1, w_2 \in W(S)$ then:
    $u = w_1 a^0 w^2 \Leftrightarrow v = w_1 w_2$
    $u = w_1 a^p a^q w^2 \Leftrightarrow v = w_1 a^{p+q} w_2$
  - If $u, v \in W(S)$ then $u \sim v$ iff one can be obtained from the other by a finite sequence of elementary expansions and contractions.

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of a group (next)

- We denote $F(S) = W(S)/_\sim$ the set of equivalence classes of words. On $F(S)$ we define the operations:
  - multiplication: $[u][v] = [uv]$ (inherited from $W(S)$);
  - inverse: $[u]^{-1}$ the word obtained from u by reversing the order of its syllables and changing the sign of each exponent.
  - $(F(S), \cdot, {}^{-1})$ is a group, called the free group on the set S.
- Intuition for the significance of the free group: a free group on a set S denoted $F(S)$ is a group in which each element can be uniquely described as a finite length product of the form $s_1^{a_1} \cdot s_2^{a_2} \cdot \ldots \cdot s_n^{a_n}$, where s_i are distinct elements of S and $a_i \in \mathbb{Z}^*, \forall i \in \{1, \ldots, n\}$. The set S is called the set of generators.
- Theorem: Any group is the homomorphic image of some free group.

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of a group (next)

- Informally , for the presentation of a group one specifies a set S of generators of G s.t. every element of the group can be written as a product of some of these generators, and a set R of relations among those generators. We say G has a presentation $G = \langle S \mid R \rangle$.
- Formally , a group G has the presentation $G = \langle S \mid R \rangle$ if it is isomorphic to the quotient of the free group $F(S)$ by the normal subgroup N of $F(S)$ generated by the relations R.
  - $G = \langle S \mid R \rangle \sim_{\textit{isomorphic}} F(S)/N$
  - To form $G = \langle S \mid R \rangle$ the idea is to take the smallest quotient of $F(S)$ s.t. each element of R gets identified with the identity element. R may not be a subgroup (normal subgroup), so we cannot take a quotient by R. We take N the normal closure of R in $F(S)$, i.e. the smallest normal subgroup in $F(S)$ which contains R.

Invariants of knots: Alexander polynomial (Fox's method)

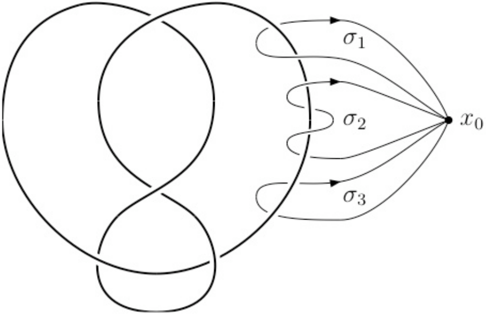
Presentation of a knot group

- A presentation of a knot group $G(K)$ (i.e. the fundamental group of the knot complement $G(K) := \pi(\mathbb{R}^3 \setminus K)$) is the Wirtinger presentation. A proof that this presentation describe $G(K)$ is found in Dale Rolfsen, "Knots and links". The proof uses the van Kampen theorem. We now describe the Wirtinger presentation of $G(K)$.
- $G(K)$ and operations on $G(K)$ (in the Wirtinger presentation)
  - an element of $G(K)$ is represented by a loop which begins at some fixed base point $x_0 \notin K$, winds through the space around K and returns to x_0.
  - the composition operation in $G(K)$ corresponds to the concatenation of loops;
  - the identity element is represented by a path that never leaves x_0, or by a loop at x_0 which never gets tangled up with any part of K s.t. it can be shrunk back to x_0 without getting caught anywhere;
  - the inverse of the group element represented by a loop σ is represented by the same path traced in the opposite direction.

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of a knot group (next)

- $G(K)$ and operations on $G(K)$ (in the Wirtinger presentation)
- Three loops in the complement of a figure-eight knot:



- If loop σ_1 represents the group element g, then σ_2 represents g^2, σ_3 represents the identity.

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of a knot group (next)

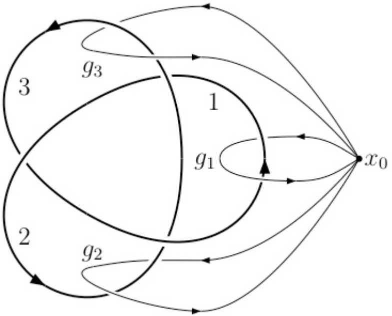
Given an oriented diagram of K, we have a "recipe" to write down a group presentation for $G(K) = \langle S \mid R \rangle$

- How do we get the generators S ?
  - we consider x_0 somewhere off to the side of the diagram. For each arc i in the diagram, we write down a group element g_i, represented by a loop which begins at x_0, crosses under arc i from right to left and then crosses over arc i and returns to x_0 without getting tangled up anywhere else in the knot.
  - all the elements g_i generate $G(K)$ with $g_i \in S$: $\forall x_0$-based loop through the space around K can be deformed into a sequence of loops each of which leaves x_0, circles one arc of K and returns to x_0.

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of a knot group (next)

- How do we get the generators S ?
- The generators of the knot group of the trefoil knot are:

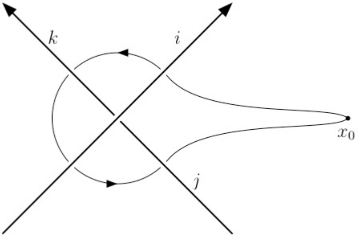


- $S = \{g_1, g_2, g_3\}$

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of a knot group (next)

- How do we get the relations R ?
  - for each crossing in the diagram, we have a relation among the generators $g_i \in S$. At the following type of crossing:

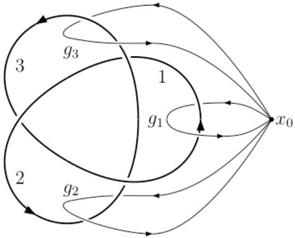


the loop which passes under all the 3 arcs and circles the crossing once represents the group element $g_i g_k g_i^{-1} g_j^{-1}$ since it can be deformed into 4 loops representing these generators. Since this loop can be pulled clear of K, it is the identity element, so we get $g_i g_k g_i^{-1} g_j^{-1} = 1$.

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of the knot group of the trefoil knot (example)

$G = \langle S \mid R \rangle = ?$

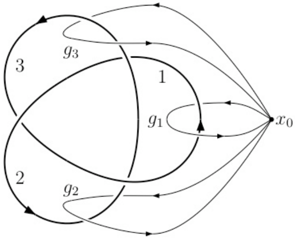


- $S = ?$
- $R = ?$

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of the knot group of the trefoil knot (example)

$G = \langle S \mid R \rangle =?$

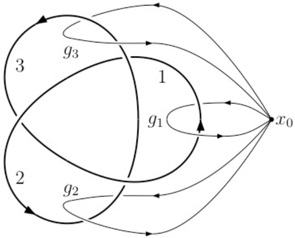


- $S =?$
  - $S = \{g_1, g_2, g_3\}$
- $R =?$

Invariants of knots: Alexander polynomial (Fox's method)

Presentation of the knot group of the trefoil knot (example)

$G = \langle S \mid R \rangle = ?$



- $S = ?$
  - $S = \{g_1, g_2, g_3\}$
- $R = ?$
  - $R = \{g_2 g_1 g_2^{-1} g_3^{-1} = 1,\ g_3 g_2 g_3^{-1} g_1^{-1} = 1,\ g_1 g_3 g_1^{-1} g_2^{-1} = 1\}$

Invariants of knots: Alexander polynomial (Fox's method)

The knot polynomials The existence and the uniqueness of the knot polynomials depend on the abelianized group of a knot group.

Definition

Let $F(\langle x_1, ..., x_n \rangle)$ be the free group of the generators $\langle x_1, ..., x_n \rangle$ and $\mathbb{Z}[F]$ the group ring of F. We define the free derivative for every generator x_i from $\langle x_1, ..., x_n \rangle$ as follows: $D_i : F(\langle x_1, ..., x_n \rangle) \to \mathbb{Z}[F], D_i := \frac{\partial}{\partial x_i}$ with the properties:

$$\frac{\partial}{\partial x_i} 1 = 0, \quad \frac{\partial x_j}{\partial x_i} = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}, \quad \frac{\partial x_j^{-1}}{\partial x_i} = -\delta_{ij} x_j^{-1},$$

and for any word $w = u x_j \in F(\langle x_1, ..., x_n \rangle)$:

$$\frac{\partial}{\partial x_i} u x_j = \frac{\partial}{\partial x_i} u + u \frac{\partial x_j}{\partial x_i}.$$

Invariants of knots: Alexander polynomial (Fox's method)

The knot polynomials The existence and the uniqueness of the knot polynomials depend on the abelianized group of a knot group.

- $\forall g, h \in G$ group, the commutator of g, h is $[g, h] = g^{-1}h^{-1}gh$. The commutator group $[G, G]$ is the subgroup of G generated by all commutators, and $[G, G]$ is a normal subgroup of G. The quotient $G/[G, G]$ is an abelian group, called abelianization of G.

Definition

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$$\frac{\partial}{\partial x_i}1 = 0, \quad \frac{\partial x_j}{\partial x_i} = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}, \quad \frac{\partial x_j^{-1}}{\partial x_i} = -\delta_{ij}x_j^{-1},$$

and for any word $w = ux_j \in F(\langle x_1, ..., x_n \rangle)$:

$$\frac{\partial}{\partial x_i}ux_j = \frac{\partial}{\partial x_i}u + u\frac{\partial x_j}{\partial x_i}.$$

Invariants of knots: Alexander polynomial (Fox's method)

The knot polynomials

- Let $F(S)$ be the free group on $S = \langle x_1, ..., x_n \rangle$, $G(K) = (\langle x_1, ..., x_n \rangle \mid \langle r_1, ..., r_n \rangle)$ a knot group presentation. Let $\phi : F(S) \to G, \phi(s) = s$ a group homomorphism (any group is a homomorphic image of some free group). Then $\phi : \mathbb{Z}[F] \to \mathbb{Z}[G]$ is a ring homomorphism.

Invariants of knots: Alexander polynomial (Fox's method)

The knot polynomials

- Let $F(S)$ be the free group on $S = \langle x_1, ..., x_n \rangle$, $G(K) = (\langle x_1, ..., x_n \rangle \mid \langle r_1, ..., r_n \rangle)$ a knot group presentation. Let $\phi : F(S) \to G, \phi(s) = s$ a group homomorphism (any group is a homomorphic image of some free group). Then $\phi : \mathbb{Z}[F] \to \mathbb{Z}[G]$ is a ring homomorphism.
- Let $\psi : G \to G^{'}$ group homomorphism with $G^{'} = \psi(G)$ abelian group (i.e. ψ is the abelianization of G). Based on the theorem below, $\psi(G)$ is an infinite cyclic group generated by one element let's choose it to be t and so $\psi : \mathbb{Z}[G] \to \mathbb{Z}[G^{'}] = \mathbb{Z}[t, t^{-1}], x_i \xrightarrow{\psi} t$ ring homomorphism.
    - Theorem: The abelianized group of every knot group is infinite cyclic (The generators of a presentation are all mapped into the same generator).
    - Theorem: The group ring of an infinite cyclic group is a *gcd* domain.
- Definition: Let G be an infinite cyclic group. Then to any knot group presentation $G(K) = (\langle x_1, ..., x_n \rangle \mid \langle r_1, ..., r_m \rangle)$ we associate a Jacobian matrix of dimension $m \times n$ called the Alexander matrix of $G(K)$ whose ij^{th} entry is $\psi\phi(\frac{\partial r_i}{\partial x_j})$ with $i \in [1, ..., m], j \in \{1, ..., n\}$.

Invariants of knots: Alexander polynomial (Fox's method)

The knot polynomials

- Definition: For any integer $k \geq 0$, the k^{th} knot polynomial Δ_k of a finite presentation $G(K) = (\langle x_1, ..., x_n\rangle | \langle r_1, ..., r_m\rangle)$ of a knot group is the *gcd* of the determinants of all $(n-k) \times (n-k)$ submatrices of the Alexander matrix of $G(K)$ where $\Delta_k = 0,$ *if* $n - k > m$ and $\Delta_k = 1,$ *if* $n - k \leq 0$.
- Definition: Let $A \in \mathcal{M}_{m\times n}(R)$ be the Alexander matrix of the knot group presentation $G(K) = (S|R)$. Then $\forall k \in \mathbb{Z}_+$ the k^{th} elementary ideal $E_k(A)$ of A is the ideal generated by the determinants of all $(n-k) \times (n-k)$ submatrices of A if $0 < n - k \leq m$ and $E_k(A) = 0, n - k > m, E_k(A) = R, n - k \geq 0$.

Invariants of knots: Alexander polynomial (Fox's method)

The knot polynomials

- Definition: For any integer $k \geq 0$, the k^{th} knot polynomial Δ_k of a finite presentation $G(K) = (\langle x_1, ..., x_n \rangle | \langle r_1, ..., r_m \rangle)$ of a knot group is the *gcd* of the determinants of all $(n-k) \times (n-k)$ submatrices of the Alexander matrix of $G(K)$ where $\Delta_k = 0,$ *if* $n - k > m$ and $\Delta_k = 1,$ *if* $n - k \leq 0$.
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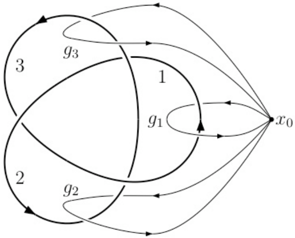
- Theorem: The 1st elementary ideal of a knot group is an invariant for the knot and does not depend on the presentation of the knot group.
- Theorem: The 1st elementary ideal of a knot group is a principal ideal generated by the 1st knot polynomial $\Delta_1(t) = \Delta(t)$ called the Alexander polynomial. It follows that the determinant of any one of the $(n-1) \times (n-1)$ submatrices of A may be taken to be Δ.

Invariants of knots: Alexander polynomial (Fox's method)

The Alexander knot polynomial of the trefoil knot (example)



- A presentation of the knot group of the trefoil knot is:

  $G = (S \mid R) = (\{g_1, g_2, g_3\} \mid \{g_2 g_1 g_2^{-1} g_3^{-1} = 1,\ g_3 g_2 g_3^{-1} g_1^{-1} = 1,\ g_1 g_3 g_1^{-1} g_2^{-1} = 1\}$

- The Alexander matrix is the Jacobian $J = \left\{(a_{ij}) = \psi\phi(\dfrac{\partial r_i}{\partial g_j})|i, j \in \{1, 2, 3\}\right\}$ with $\phi : \mathbb{Z}[F] \to \mathbb{Z}[G], \phi(s) = s$, and $\psi : \mathbb{Z}[G] \to \mathbb{Z}[t, t^{-1}], \psi(g_i) = t, i \in \{1, 2, 3\}$.

Invariants of knots: Alexander polynomial (Fox's method)

The Alexander knot polynomial of the trefoil knot (example)

- $G = (\{g_1, g_2, g_3\} | \{r_1 := g_2 g_1 g_2^{-1} g_3^{-1} = 1, r_2 := g_3 g_2 g_3^{-1} g_1^{-1} = 1, r_3 := g_1 g_3 g_1^{-1} g_2^{-1} = 1\})$
- $$J = \left(\begin{array}{c|ccc} & g_1 & g_2 & g_3 \\ \hline \dfrac{\partial r_1}{g_i} & \psi(g_2) & \psi(1 - g_2 g_1 g_2^{-1}) & \psi(-g_2 g_1 g_2^{-1} g_3^{-1}) \\ \dfrac{\partial r_2}{g_i} & \psi(-g_3 g_2 g_3^{-1} g_1^{-1}) & \psi(g_3) & \psi(1 - g_3 g_2 g_3^{-1}) \\ \dfrac{\partial r_3}{g_i} & \psi(1 - g_1 g_3 g_1^{-1}) & \psi(-g_1 g_3 g_1^{-1} g_2^{-1}) & \psi(g_1) \end{array}\right)$$
- $$J = \begin{pmatrix} t & 1-t & -1 \\ -1 & t & 1-t \\ 1-t & -1 & t \end{pmatrix}$$
- The 2×2 minors generate the elementary ideal $\langle t^2 - t + 1 \rangle$, which is a principal ideal generated by the Alexander polynomial $\Delta_K(t) = t^2 - t + 1$.

Invariants of knots: Alexander polynomial (Conway's method)

J. Conway (1969) became famous for discovering a polynomial which satisfy a skein relation. Unfortunately, it turned out that he had rediscovered the Alexander polynomial, plus Alexander himself mentioned the same method in his original paper. Still, Conway's paper prepared the discovery of the Jones polynomial 15 years later.

Definition

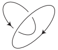
A skein relation is an equation that relates the polynomial of a link to the polynomials of links obtained by changing the crossings in a diagram of the original link.

Invariants of knots: Alexander polynomial (Conway's method)

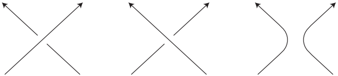
Conway showed that the Alexander polynomial can be computed using just three rules (axioms).

- Rule 1: To each oriented knot/link there is associated a polynomial $\Delta(K) \in \mathbb{Z}[z]$. Equivalent knots/links receive identical polynomials: $K \sim K^{'} \Rightarrow \Delta(K) = \Delta(K^{'})$.
- Rule 2:

$\Delta($  $) = 1$

$\Delta($  $) = 0$

- Rule 3: Suppose that 3 knots/links differ at the arcs of one crossing as show below:



$L+$ $L-$ L_0

then

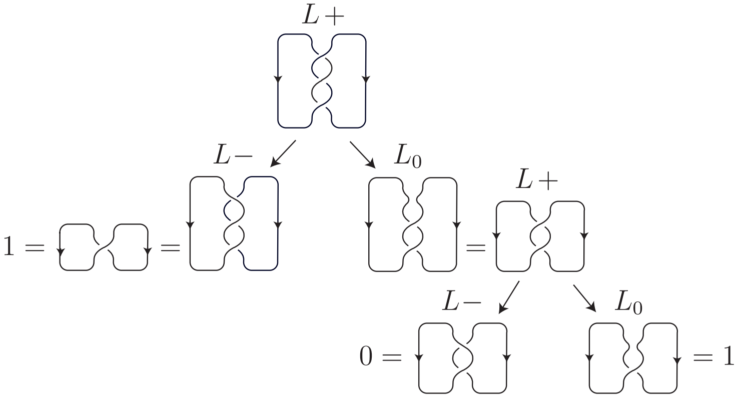
$$\Delta(L_+) = \Delta(L_-) - z\Delta(L_0) \textit{(skein relation.)}$$

Invariants of knots: Alexander polynomial (Conway's method)

- It can be proven that these rules are enough to ensure that the Alexander polynomial is an invariant for knots and links (Conway's achievement). A thoroughly description of the proof can be found in Kauffman's paper "On knots".
- Given a diagram, we choose a crossing point to be the crossing point we change in order to turn the projection into the trivial projection. This process of repeatedly choosing a crossing, and then applying the skein relation to obtain 2 simpler links yields a tree of links called the resolving tree. Conway polynomial is a polynomial in z, where $z = t^{1/2} - t^{-1/2}$.

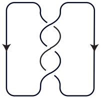
Invariants of knots: Alexander polynomial (Conway's method)

Resolving tree of the trefoil: use recursively the skein relation
$\Delta(L_+) = \Delta(L_-) - z\Delta(L_0)$



Invariants of knots: Alexander polynomial (Conway's method)

The Conway polynomial for the trefoil knot using recursively the skein relation $\Delta(L_+) = \Delta(L_-) - z\Delta(L_0)$:

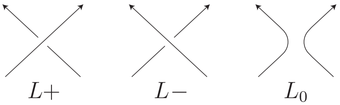
$\Delta($  $) = 1 - z(0 - z) = 1 - z(-z) = 1 + z^2$

Replacing $z = t^{1/2} - t^{-1/2}$ we get $\Delta_{trefoil}(t) = t^2 - t + 1$.

Invariants of knots: Jones polynomial

The Jones polynomial computed using the following rules is an invariant for knots:

- Rule 1: $V_{unknot}(t) = 1$
- Rule 2: Suppose that 3 knots/links differ at the arcs of one crossing as show below:



then:

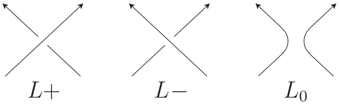
$$t^{-1} * V(L_+) - t * V(L_-) - (t^{1/2} - t^{-1/2}) * V(L_0) = 0$$

Example: $V_{trefoil}(t) = -t^4 + t^3 + t$

Invariants of knots: HOMFLY polynomial

The HOMFLY polynomial computed with the following rules is an invariant for knots:

- Rule 1: $P_{unknot}(l, m) = 1$
- Rule 2: Suppose that 3 knots/links differ at the arcs of one crossing as show below:



then:

$$l * P(L_+) + l^{-1} * P(L_-) + m * P(L_0) = 0$$

- The HOMFLY polynomial is a two-variable Laurent polynomial, the variables being l and m, i.e. $P(L_+) \in [Z]l^{\pm 1}, m^{\pm 1}$.
- Substitution for Alexander polynomial: $l = i,\ m = i * (t^{1/2} - t^{-1/2})$
- Substitution for Jones polynomial: $l = i * t^{-1},\ m = i * (t^{-1/2} - t^{1/2})$

1 Motivation

2 Fundamentals of Knot Theory

Knots and equivalence of knots

Diagrams of knots

Types of knots

3 Advanced topics in knot theory

Reidemeister moves and theorem

Invariants of knots/links

Tabulation of knots

4 Modern aspects of knot theory

5 Conclusion

Tabulation of knots

Type	Achievements	Purpose
classification	complete table only up to prime knots with 13 crossings	complete knot table
knot invariants	distinguish certain types of knots	complete set of knot invariants

Remark: Some knot tables are made up using Dowker notation, which was not presented in this lecture.

Modern aspects of knot theory

- At present, knot theory is still a dynamic branch of topology. It has concrete applications in the study of enzymes acting on DNA strands.
- DNA is tangled up in knots. DNA are long, thin molecules found inside the nucleus of a cell. Imagine the nucleus as a basketball, and inside it a thin fishing line with 200 km, packed inside, which is the DNA.
- DNA must be topologically manipulated in order for vital process to take place. DNA must first unpack itself so that it can interact with enzymes. By thinking of DNA as a knot, we can use knot theory to estimate how hard DNA is to unknot.

Conclusion

- we tried to present and formalize some of the notions and results of knot theory (a lot of aspects were still not presented such as braids, or rational tangles).
- knot theory is a dynamic area of topology, with a lot of open problems, which represent a challenge for mathematicians.

Thank you for your attention.