A symbolic-numeric algorithm for genus computation

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

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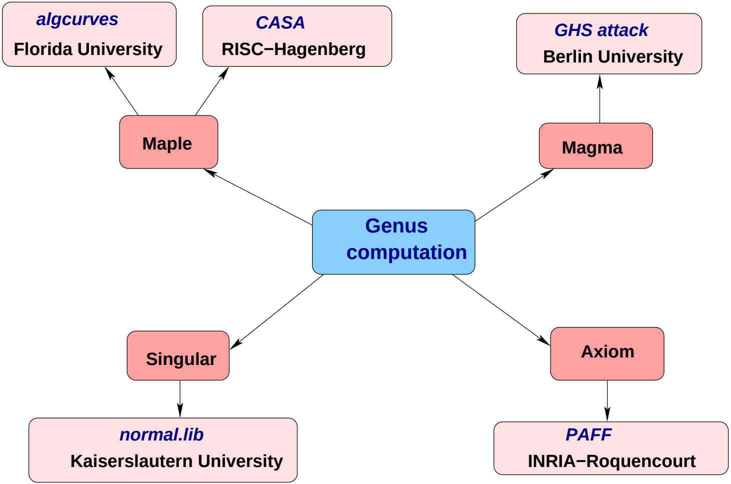
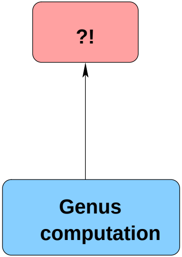
Exact Algorithms:

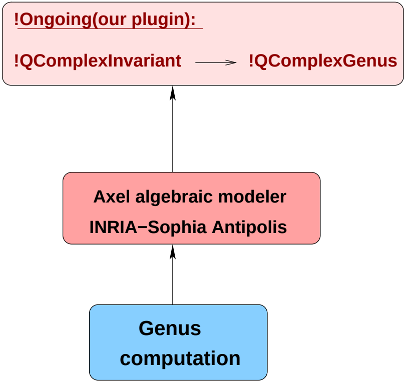


Numeric Algorithms:



Numeric Algorithms:

DK Project: Symbolic-Numeric techniques for genus computation and parametrization (project leader: Prof. Dr. Josef Schicho).



1. Motivation
2. Describing the Problem
   - What?
3. Solving the problem
   - How?
4. Current results
5. Conclusion and future work

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What?

- Input:
  - $\mathbb{C}$ field of complex numbers;
  - $F \in \mathbb{C}[z, w]$ irreducible with coefficients of limited accuracy [1];
  - $C = \{(z, w) \in \mathbb{C}^2 | F(z, w) = 0\} =$
    $= \{(x, y, u, v) \in \mathbb{R}^4 | F(x + iy, u + iv) = 0\}$ complex algebraic curve
    (d is the degree, $Sing(C)$ is the set of singularities);
- Output:
  - approximate $genus(C)$ s.t.

$$genus(C) = \frac{1}{2}(d-1)(d-2) - \sum_{P \in Sing(C)} \delta\text{-invariant}(P);$$

[1] For now: symbolic coefficients

1. Motivation

2. Describing the Problem
   - What?

3. Solving the problem
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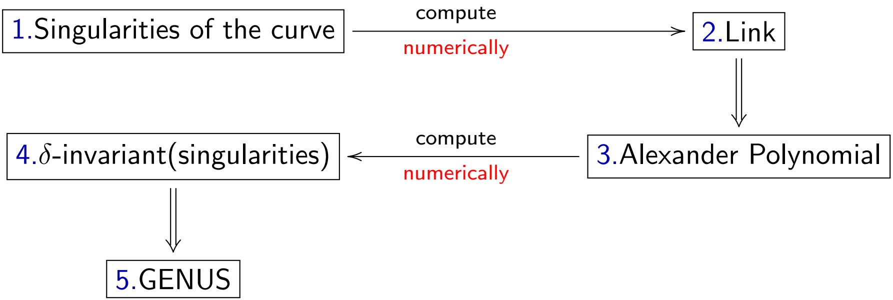
4. Current results

5. Conclusion and future work

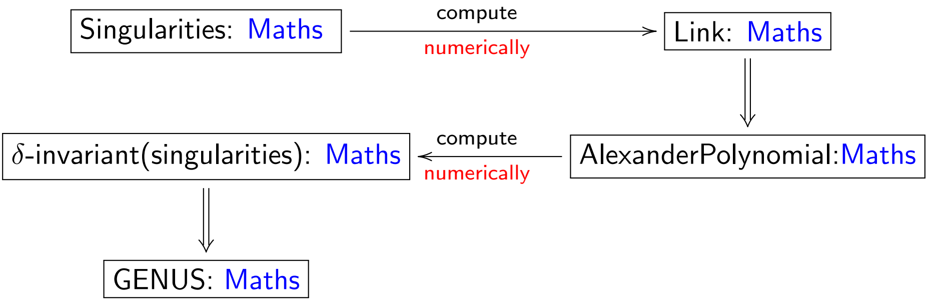
How?

- Strategy for computing the genus



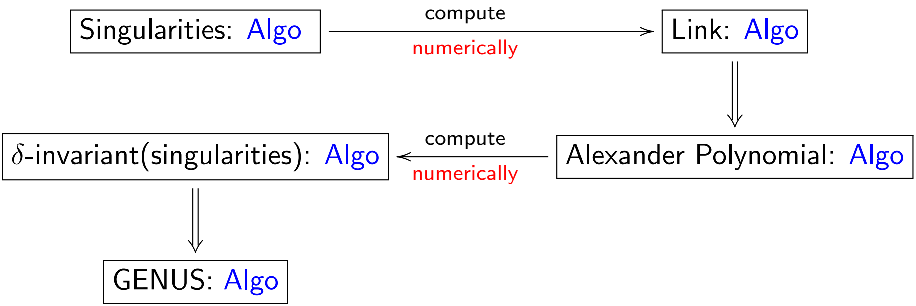
How?

- Method for computing the genus



How?

- Algorithm for the method



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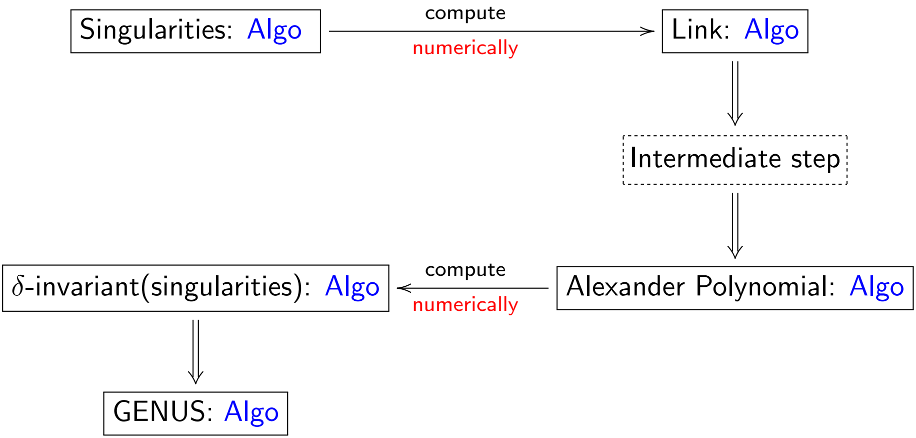
How?

- Algorithm for the method



Solving the problem

Implementation of the algorithm

- (*Mathematica* computer algebra system)
- *Axel* algebraic geometric modeler [a]
  - developed by *Galaad* team (INRIA Sophia-Antipolis);
  - written in Qt scripting language;
  - topology of implicit curves;
  - intersections of implicit surfaces.

[a] Acknowledgements: B. Mourrain, J. Wintz

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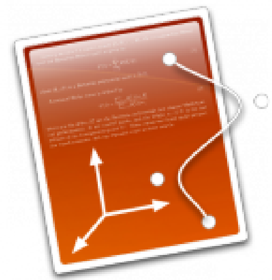

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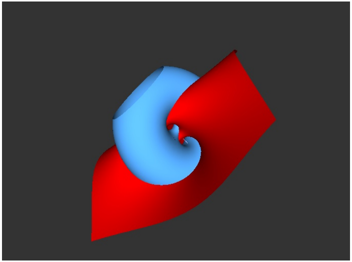

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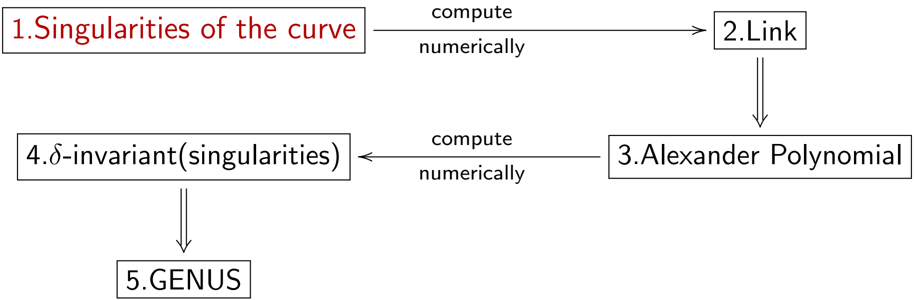
First


1.Singularities of the curve
compute
numerically
2.Link
3.Alexander Polynomial
compute
numerically
4.δ-invariant(singularities)
5.GENUS

Computing the singularities of the curve

- Input:
  - $F \in \mathbb{C}[z, w]$
  - $C = \{(z, w) \in \mathbb{C}^2 | F(z, w) = 0\}$
- Output:
  - $S = \{(z_0, w_0) \in \mathbb{C}^2 | F(z_0, w_0) = 0, \frac{\delta F}{\delta z}(z_0, w_0) = 0, \frac{\delta F}{\delta w}(z_0, w_0) = 0\}$

Method: $\Rightarrow$ solve overdeterminate system of polynomial equations in $\mathbb{C}^2$:

$$\begin{cases} F(z_0, w_0) = 0 \\ \frac{\delta F}{\delta z}(z_0, w_0) = 0 \\ \frac{\delta F}{\delta w}(z_0, w_0) = 0 \end{cases}, \tag{1}$$

Computing the singularities of the curve

or in $\mathbb{R}^4$: $F(z,w) = F(x+iy, u+iv) = s(x,y,u,v) + it(x,y,u,v)$

$$\begin{cases} s(x_0,y_0,u_0,v_0) = 0 \\ t(x_0,y_0,u_0,v_0) = 0 \\ \\ \dfrac{\delta s}{\delta x}(x_0,y_0,u_0,v_0) = 0 \\ \\ \dfrac{\delta t}{\delta x}(x_0,y_0,u_0,v_0) = 0 \\ \\ \dfrac{\delta s}{\delta u}(x_0,y_0,u_0,v_0) = 0 \\ \\ \dfrac{\delta t}{\delta u}(x_0,y_0,u_0,v_0) = 0 \end{cases} , \tag{2}$$

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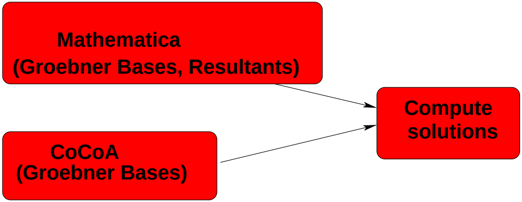
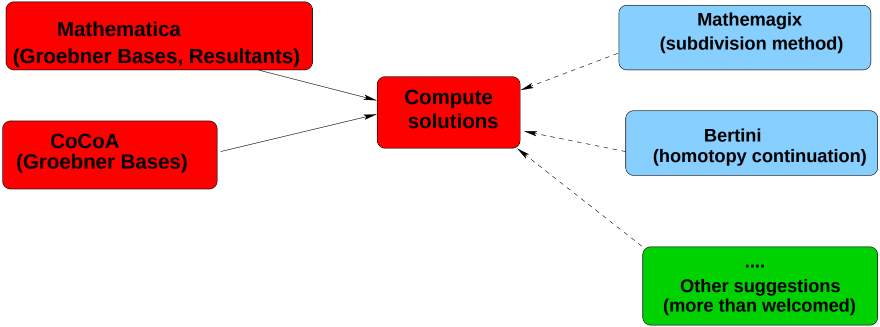
Computing the singularities of the curve

Using numeric input polynomials



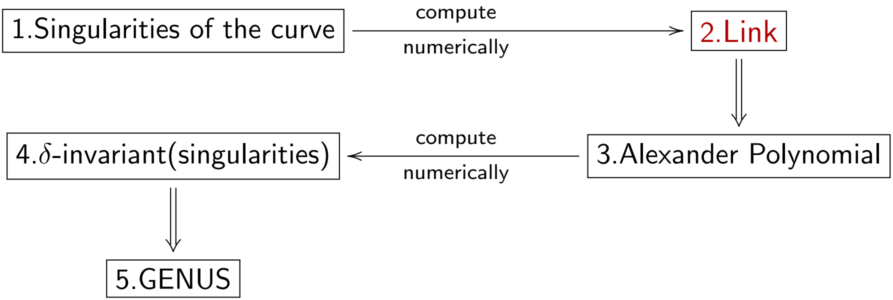
Computing the singularities of the curve

Using numeric input polynomials



Note: so far an open problem.

Next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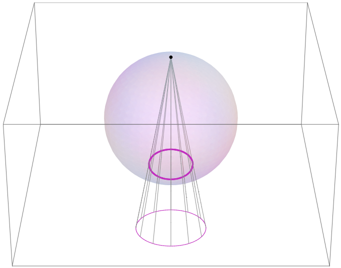
Computing the link of the singularity

- Why the link of a singularity?
  - helps in understanding the topology of a complex curve near a singularity;
- How do we compute the link?
  - use stereographic projection;



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Computing the link of the singularity

Method (based on Milnor's results)

1. Let $C = \{(x,y,u,v) \in \mathbb{R}^4 | F(x+iy, u+iv) = 0\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$, with $\left(F(0,0), \frac{\delta F}{\delta z}(0,0), \frac{\delta F}{\delta w}(0,0)\right) = (0,0,0)$, where $z = x+iy, w = u+iv.$

2. Consider $S^3 = \{(x,y,u,v) \in \mathbb{R}^4 | x^2+y^2+u^2+w^2 = \epsilon^2\} \subset \mathbb{R}^4$ and $X = C \bigcap S^3 = \{(x,y,u,v) \in \mathbb{R}^4 | F(x,y,u,v) = 0, x^2+y^2+u^2+w^2 = \epsilon^2\}.$

3. For $P(0,0,0,\epsilon) \in S^3 \setminus C$, construct
$f : S^3 \setminus \{P\} \subset \mathbb{R}^4 \to \mathbb{R}^3, (x,y,u,v) \to (a,b,c) = (\frac{x}{\epsilon - v}, \frac{y}{\epsilon - v}, \frac{u}{\epsilon - v})$
$f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) \in C \bigcap S^3 : (a,b,c) = f(x,y,u,v)\}$
$f(X)$ is a link.

Computing the link of the singularity

Method (next)

3. $f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) \in C \bigcap S^3 : (a,b,c) = f(x,y,u,v)\}$

$$f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) = f^{-1}(a,b,c) \in C \bigcap S^3\}$$

4. Compute $f^{-1} : \mathbb{R}^3 \to S^3 \setminus \{P\}$

$$(a,b,c) \to (x,y,u,v) = \left(\frac{2a\epsilon}{1+a^2+b^2+c^2}, \frac{2b\epsilon}{1+a^2+b^2+c^2}, \frac{2c\epsilon}{1+a^2+b^2+c^2}, \frac{\epsilon(a^2+b^2+c^2-1)}{1+a^2+b^2+c^2}\right)$$

5. Get $\begin{array}{l} f(X) = \Big\{(a,b,c) \in \mathbb{R}^3 | \\ F\left(\frac{2a\epsilon}{1+a^2+b^2+c^2}, \frac{2b\epsilon}{1+a^2+b^2+c^2}, \frac{2c\epsilon}{1+a^2+b^2+c^2}, \frac{\epsilon(a^2+b^2+c^2-1)}{1+a^2+b^2+c^2}\right) = 0\Big\} \end{array} \Leftrightarrow$

$$f(X) = \{(a,b,c) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$$

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Computing the link of the singularity

Method (next)

3. $f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) \in C \bigcap S^3 : (a,b,c) = f(x,y,u,v)\}$
$$f(X) = \{(a,b,c) \in \mathbb{R}^3 | \exists (x,y,u,v) = f^{-1}(a,b,c) \in C \bigcap S^3\}$$

4. Compute $f^{-1} : \mathbb{R}^3 \to S^3 \setminus \{P\}$
$$(a,b,c) \to (x,y,u,v) = \left(\frac{2a\epsilon}{1+a^2+b^2+c^2}, \frac{2b\epsilon}{1+a^2+b^2+c^2}, \frac{2c\epsilon}{1+a^2+b^2+c^2}, \frac{\epsilon(a^2+b^2+c^2-1)}{1+a^2+b^2+c^2}\right)$$

5. Get
$$f(X) = \Big\{(a,b,c) \in \mathbb{R}^3 | \; F\left(\frac{2a\epsilon}{1+a^2+b^2+c^2}, \frac{2b\epsilon}{1+a^2+b^2+c^2}, \frac{2c\epsilon}{1+a^2+b^2+c^2}, \frac{\epsilon(a^2+b^2+c^2-1)}{1+a^2+b^2+c^2}\right) = 0\Big\}$$

Compute B s.t.
$$f(X) = \{(a,b,c) \in B \subset \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$$ is a link

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Computing the link of the singularity

Method (next)

6. For $f(X) = \left\{(a,b,c) \in B \subset \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\right\}$ link

find $B = max\left\{||f(p)||_\infty, p \in S^3 \bigcap C\right\} \leq max\{||f(p)||_2, p \in S^3 \bigcap C\}$

...

7. Compute

$$v_0 = max\{v : (x,y,u,v) \in S^3 \bigcap C\} \text{ s.t. } v \text{ is solution for}$$

$$\begin{cases} x^2 + y^2 + u^2 + v^2 - \epsilon^2 = 0 \\ ReF(x+iy, u+iv) = 0 \\ ImF(x+iy, u+iv) = 0 \end{cases},$$

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Computing the link of the singularity

Method (next)

6. For $f(X) = \{(a,b,c) \in B \subset \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$ link

find $B = max\{||f(p)||_\infty, p \in S^3 \bigcap C\} \leq max\{||f(p)||_2, p \in S^3 \bigcap C\}$
...

7. Compute

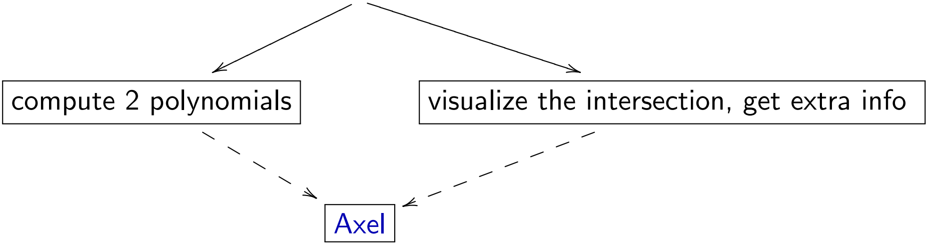
$$b = \sqrt{\frac{\epsilon + v_0}{\epsilon - v_0}}$$

Get $B = [-b, b]^3$

Computing the link of the singularity

Method (summary)

$$f(X) = \{(a,b,c) \in B \subset \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$$



Computing the link of the singularity

Why Axel?

Axel computes the topology of implicit curves in $\mathbb{R}^3$.

In our case:

- Input:
  - $ReF(...), ImF(...) \in \mathbb{R}[a, b, c]$
  - $C = \{(a, b, c) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$
  - $B = [-b, b] \times [-b, b] \times [-b, b], \epsilon \geq 0$
- Output:
  - $Graph(C) = \langle \mathcal{V}, \mathcal{E} \rangle$ with
    $\mathcal{V} = \{p = (m, n, q) \in \mathbb{R}^3\}$
    $\mathcal{E} = \{(i, j) | i, j \in \mathcal{V}\}$
- s.t. $Graph(C) \cong_{isotopic} C$

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Computing the link of the singularity

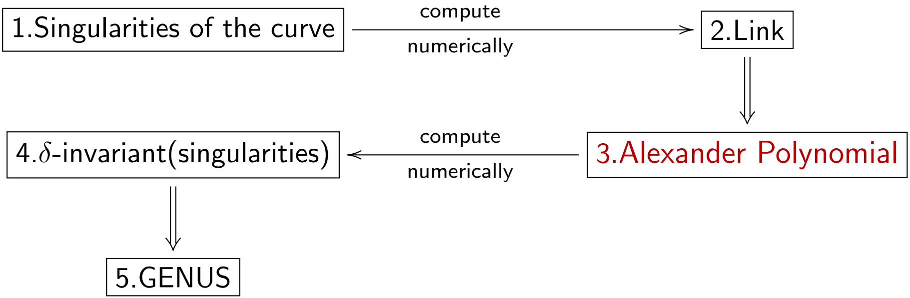
Test experiments (with Axel)

Equation	Tests on ϵ					
	ϵ=0.5		ϵ=1.0		ϵ=4.3	
	$[-b,b]^3$	link	$[-b,b]^3$	link	$[-b,b]^3$	link
$z^2 - w^2$	2.41421	**Hopf link**	2.41421	Hopf link	2.41421	Hopf link
$z^2 - w^3$	3.38298	**Trefoil knot**	2.67567	Trefoil knot	1.84639	Trefoil knot
$z^2 - w^2 - w^3$	2.37636	**Hopf link**	2.28464	Curve one singularity	2.24247	Trefoil knot

V.I. Arnold's results: $Top(z^2 - w^2 - w^3) \cong Top(z^2 - w^2)$

Note: solved problem.

Next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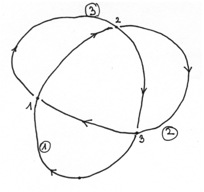
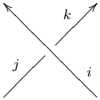
Computing the Alexander polynomial of the link



$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_1 & +1 & 1 & 2 & 3 \\ & & & & \\ & & & & \end{array} \right)$$

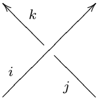
$$P(L) = \left(\quad \right)$$

Lefthanded (-1)



i	j	k
1-t	t	-1

Righthanded (+1)



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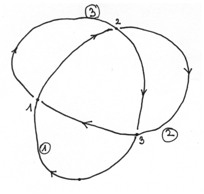

$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_1 & +1 & 1 & 2 & 3 \\ & & 1-t & -1 & t \\ & & & & \end{array}\right)$$

$$P(L) = \begin{pmatrix} 1 & 2 & 3 \\ 1-t & -1 & t \\ & & \end{pmatrix}$$

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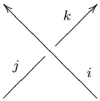

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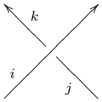

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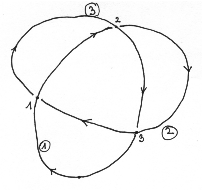
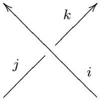
Computing the Alexander polynomial of the link



$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_2 & +1 & 3 & 1 & 2 \end{array} \right)$$

$$P(L) = \left(\quad \right)$$

Lefthanded (-1)



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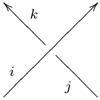

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1-t	-1	t

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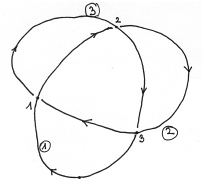
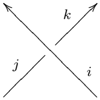


$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline & & & & \\ c_2 & +1 & 3 & 1 & 2 \\ & & 1-t & -1 & t \end{array} \right)$$

$$P(L) = \begin{pmatrix} 3 & 1 & 2 \\ 1-t & -1 & t \\ & & \end{pmatrix}$$

Lefthanded (-1)



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1-t	t	-1

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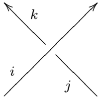

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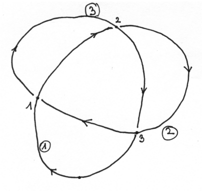
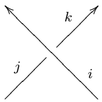


$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_2 & \color{blue}{+1} & 3 & 1 & 2 \\ & & \color{blue}{1-t} & \color{blue}{-1} & \color{blue}{t} \end{array}\right)$$

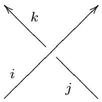
$$P(L) = \begin{pmatrix} \color{blue}{1} & \color{blue}{2} & \color{blue}{3} \\ -1 & t & 1-t \end{pmatrix}$$

Lefthanded (-1)



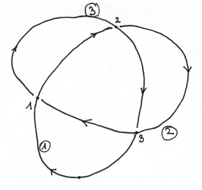
i	j	k
1-t	t	-1

Righthanded (+1)

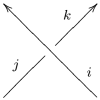


i	j	k
1-t	-1	t

Computing the Alexander polynomial of the link

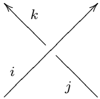


Lefthanded (-1)



i	j	k
1-t	t	-1

Righthanded (+1)

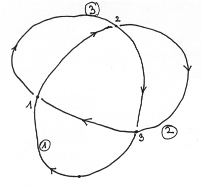


i	j	k
1-t	-1	t

$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline & & & & \\ c_3 & +1 & 2 & 3 & 1 \end{array} \right)$$

$$P(L) = \left(\quad \right)$$

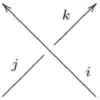
Computing the Alexander polynomial of the link



$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline & & & & \\ c_3 & +1 & 2 & 3 & 1 \\ & & 1-t & -1 & t \end{array}\right)$$

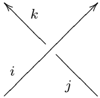
$$P(L) = \left(\begin{array}{ccc} & & \\ 2 & 3 & 1 \\ 1-t & -1 & t \end{array}\right)$$

Lefthanded (-1)



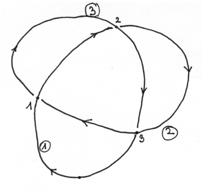
i	j	k
1-t	t	-1

Righthanded (+1)



i	j	k
1-t	-1	t

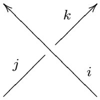
Computing the Alexander polynomial of the link



$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline & & & & \\ c_3 & +1 & 2 & 3 & 1 \\ & & 1-t & -1 & t \end{array} \right)$$

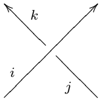
$$P(L) = \left(\begin{array}{ccc} & & \\ 1 & 2 & 3 \\ t & 1-t & -1 \end{array} \right)$$

Lefthanded (-1)



i	j	k
1-t	t	-1

Righthanded (+1)



i	j	k
1-t	-1	t

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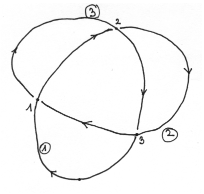
Numerical Analysis and Symbolic Computation

Computing the Alexander polynomial of the link

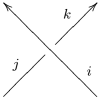


$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_1 & +1 & 1 & 2 & 3 \\ c_2 & +1 & 3 & 1 & 2 \\ c_3 & +1 & 2 & 3 & 1 \end{array}\right)$$

$$P(L) = \begin{pmatrix} 1-t & -1 & t \\ -1 & t & 1-t \\ t & 1-t & -1 \end{pmatrix}$$

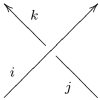
$$\Delta(L) := \Delta(t) = det(P(M)) = t^2 - t + 1$$

Lefthanded (-1)



i	j	k
1-t	t	-1

Righthanded (+1)



i	j	k
1-t	-1	t

Computing the Alexander polynomial of the link

- Input:
  - $L = K_1 \cup ... \cup K_m$ with n - crossings
  - $D(L)$- oriented diagram of L
- Output:
  - $\Delta_L(t_1, ...t_m) \in \mathbb{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$
- Method: consists of several steps

- Need D(L)!

Computing the Alexander polynomial of the link

- Input:
  - $L = K_1 \cup ... \cup K_m$ with n - crossings
  - $D(L)$- oriented diagram of L
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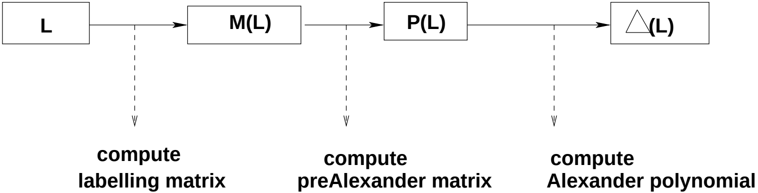
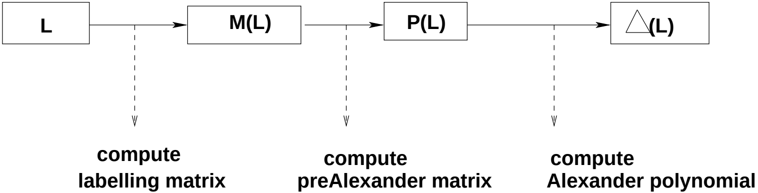
  - $\Delta_L(t_1, ...t_m) \in \mathbb{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$
- Method: consists of several steps



- Need D(L)!

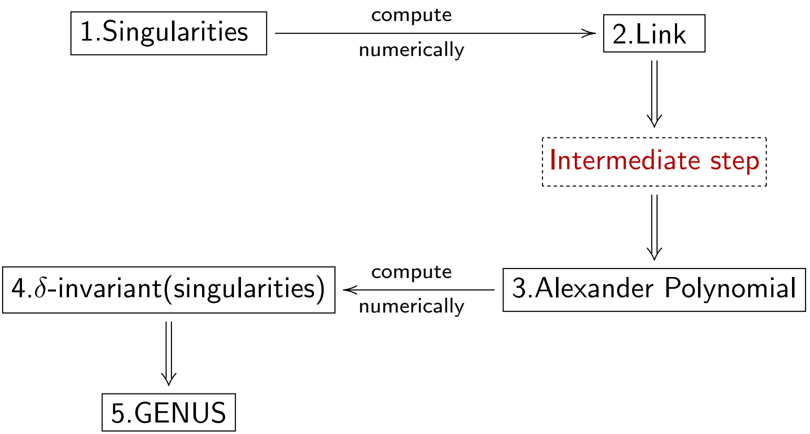
Computing the Alexander polynomial of the link

- Input:
  - $L = K_1 \cup \ldots \cup K_m$ with n - crossings
  - $D(L)$- oriented diagram of L
- Output:
  - $\Delta_L(t_1, \ldots t_m) \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$
- Method: consists of several steps



- Need D(L)!

Next


1.Singularities
compute
numerically
2.Link
Intermediate step
3.Alexander Polynomial
compute
numerically
4.δ-invariant(singularities)
5.GENUS

Intermediate step

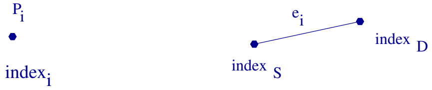
- Input:
  - $Graph(L) = \langle \mathcal{V}, \mathcal{E} \rangle$ with
    $\mathcal{V} = \{p = (m, n, q) \in \mathbb{R}^3\}$
    $\mathcal{E} = \{(i, j) | i, j \in \mathcal{V}\}$
- Output:
  - $D(L)$ s.t.
    $D(L)$ is the image under regular projection of L together with the information on each crossing telling which branch goes under and which goes over.

Intermediate step

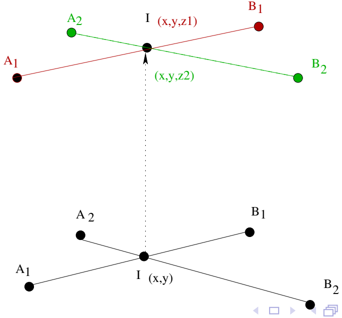
- Input:
  - $Graph'(L) = \langle \mathcal{V}', \mathcal{E}' \rangle$ with
    $\mathcal{V}' = \{p = (m, n) \in \mathbb{R}^2\}$
    $\mathcal{E}' = \{(i, j) | i, j \in \mathcal{V}'\}$
- Output:
  - $D(L)$ s.t. $D(L)$ is the image under regular projection of L together with the information on each crossing telling which branch goes under and which goes over.
- Method: the Bentley-Ottman algorithm

Intermediate step

- Input:
  - $E = \{e_1, e_2, ..., e_n\}$-set of n edges in the plane with:

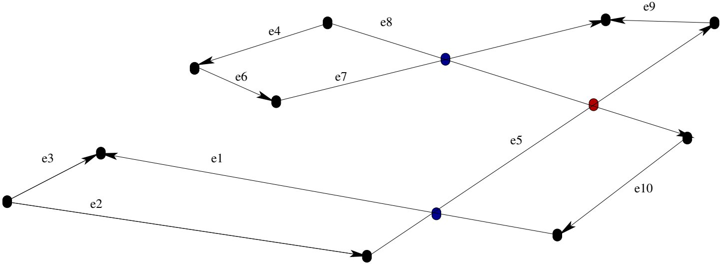


- Output:
  - $I = \{p_i = (a_i, b_i) \in \mathbb{R}^2 \mid \forall p_i \; \exists e_j \neq e_k : e_j \cap e_k = \{p_i\}\}$
    $E_I = \{(e_i, e_j) \mid i \neq j, e_i \cap e_j \neq \emptyset, z(e_i) < z(e_j))\}$



Intermediate step

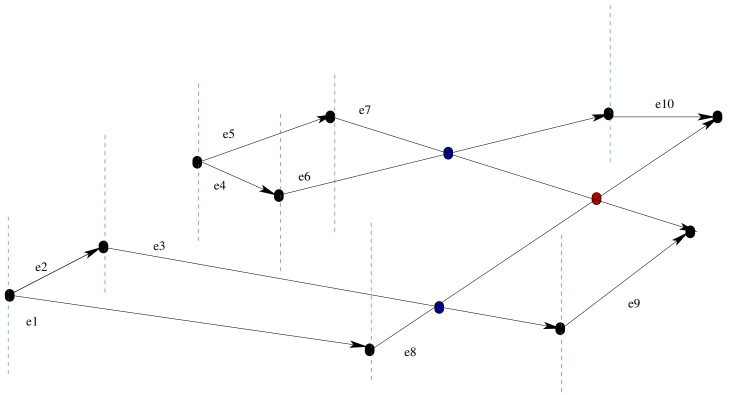
Given:



$$E = \{e_2, e_3, e_1, e_6, e_4, e_7, e_8, e_5, e_9, e_{10}\}$$

Intermediate step

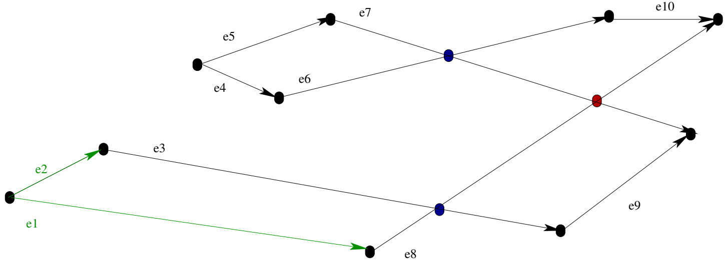
Sorting the edges-necessary condition!



$$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$$

Intermediate step

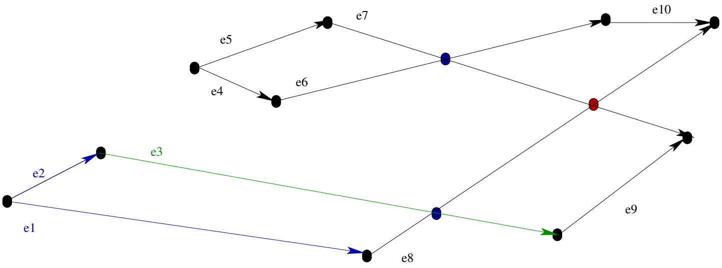
Initialization:



$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$
$SW_{list} = \{e_1, e_2\}$

Intermediate step

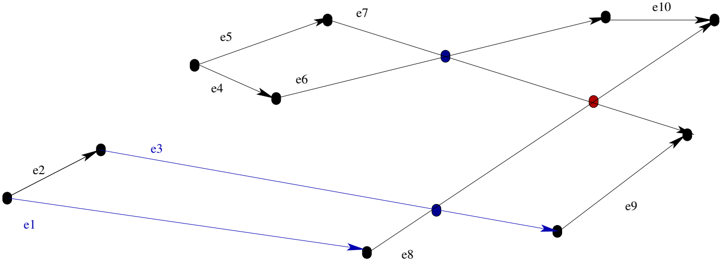
Step 1:



$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$
$SW_{list} = \{e_1, e_2\}$

Intermediate step

Step 1:



$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$

$SW_{list} = \{e_1, e_3\}$

Test $e_3 \cap e_1$? No!

$I = \emptyset$

$E_I = \emptyset$

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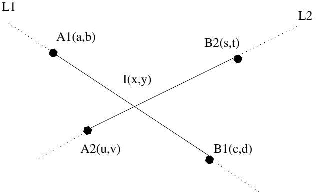
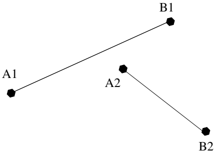
Intermediate step

How do we test intersection of 2 edges?

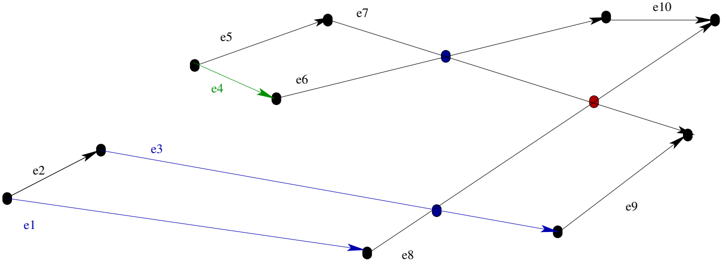




- $L_1(x,y) : y_1 = m_1 \cdot x + b_1$
  $m_1 = \dfrac{d-b}{c-a}, b_1 = \dfrac{b \cdot c - a \cdot d}{c-a}$
  $L_1(x,y) : (b-d)x + (c-a)y + a \cdot d - b \cdot c = 0$
  $L_2(x,y) : (v-t)x + (s-u)y + u \cdot t - v \cdot s = 0$
  (similarly to L_1)
- Note:
  - $L_1(A_2) \cdot L_1(B_2) < 0 \Rightarrow e_1 \cap e_2 \neq \emptyset$
    $L_2(A_1) \cdot L_2(B_1) < 0 \Rightarrow e_1 \cap e_2 \neq \emptyset$
  - $L_1(A_2) \cdot L_1(B_2) > 0 \Rightarrow e_1 \cap e_2 = \emptyset$
    $L_2(A_1) \cdot L_2(B_1) > 0 \Rightarrow e_1 \cap e_2 = \emptyset$
  - $L_1(A_2) \cdot L_1(B_2) = 0 \Rightarrow e_1 \cap e_2 = \{A_2\}$,
    $L_2(A_1) \cdot L_2(B_1) = 0 \Rightarrow e_1 \cap e_2 = \{A_2\}$,
    $(A_2 = B_1)$

Intermediate step

Step 2:



$$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$$
$$SW_{list} = \{e_1, e_3\}$$

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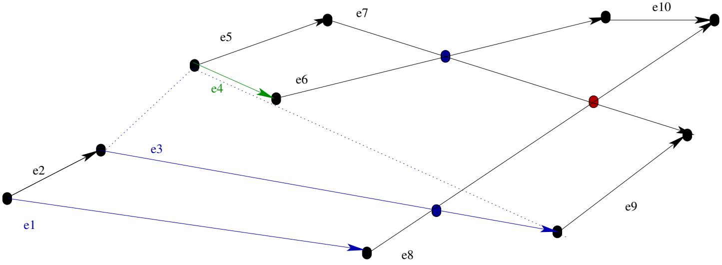
Intermediate step

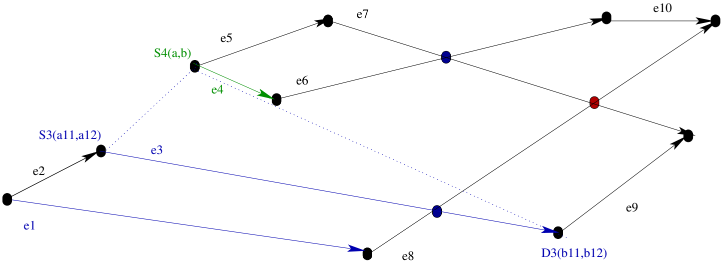
Step 2:



$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$
$SW_{list} = \{e_1, e_3\}$

Intermediate step

Step 2:



$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$
$SW_{list} = \{e_1, e_3\}$; compute:

$$det(e_4, e_3) = \begin{pmatrix} a_{11} & a_{12} & 1 \\ b_{11} & b_{12} & 1 \\ a & b & 1 \end{pmatrix} > 0 \Rightarrow e_4 \text{ after } e_3 \text{ in } SW_{list}$$

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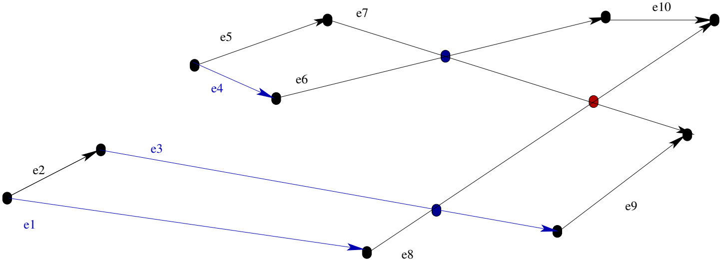
Intermediate step

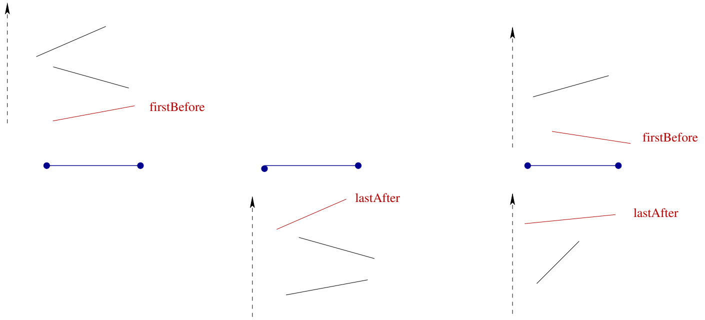
Step 2:



$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$
$SW_{list} = \{e_1, e_3, e_4\}$
Test $e_4 \cap e_3$? No!
$I = \emptyset$
$E_I = \emptyset$

Intermediate step

Keeping the sweep events list ordered - necessary condition!

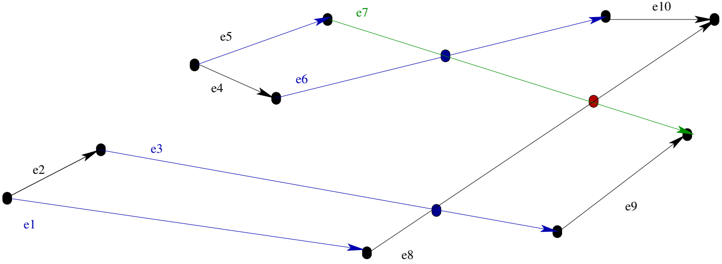


Given e, if $\forall e_i : det(e, e_i) < 0 \Rightarrow e$ before e_i
Given e, if $\forall e_i : det(e, e_i) > 0 \Rightarrow e$ after e_i

Intermediate step

....

Step 5:

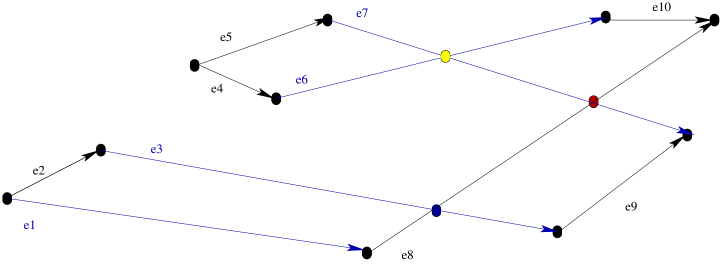


$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$

$SW_{list} = \{e_1, e_3, e_6, e_5\}$

Intermediate step

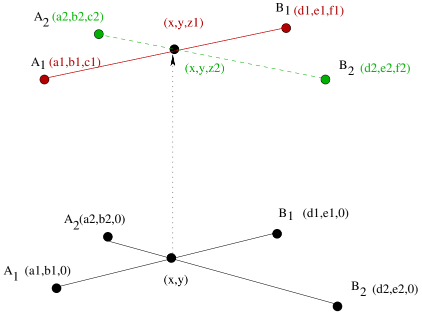
Step 5:



$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$
$SW_{list} = \{e_1, e_3, e_6, e_7\}$
Test $e_7 \cap e_6$? Yes! $\Rightarrow$
$SW_{list} = \{e_1, e_3, e_7, e_6\}$
$I = \{(a_1, b_1)\}$,
$E_I = \{(e_6, e_7)\}$

Intermediate step

Managing the info on each crossing in $\mathbb{R}^3$:



For:

$$L_2(A_1) = \begin{pmatrix} a_2 & b_2 & 1 \\ d_2 & e_2 & 1 \\ a_1 & b_1 & 1 \end{pmatrix}$$

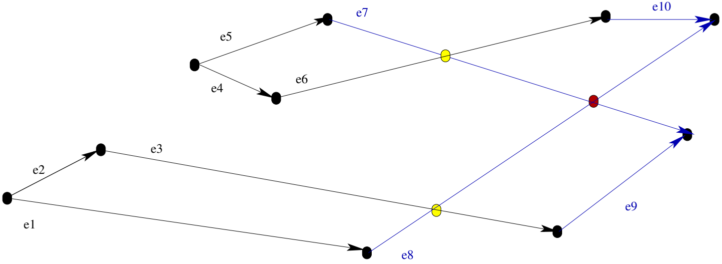
$$L_2(B_1) = \begin{pmatrix} a_2 & b_2 & 1 \\ d_2 & e_2 & 1 \\ d_1 & e_1 & 1 \end{pmatrix}$$

- Compute α_1 from:
  $\alpha_1 \cdot L_2(A_1) + (1 - \alpha_1) \cdot L_2(B_1) = 0$,
  Compute z_1 from:
  $z_1 = \alpha_1 \cdot c_1 + (1 - \alpha_1) \cdot f_1$
  (similarly compute z_2)
- Note: compare $z_1 ? z_2$
  - $z_1 > z_2 \Rightarrow e_1$ over e_2
  - $z_1 < z_2 \Rightarrow e_1$ under e_2

Intermediate step

....

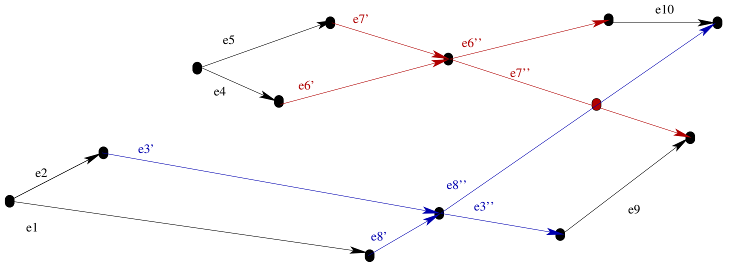
Final step:



$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$
$SW_{list} = \{e_9, e_8, e_7, e_{10}\}$
$I = \{(a_1, b_1), (a_2, b_2)\}$
$E_I = \{(e_6, e_7), (e_8, e_3)\}$

Intermediate step

Refinements of the algorithm:



Everytime an intersection is detected we update E, SW_{list} as follows:

Detect $e_6 \cap e_7$:

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_7^{''}, e_6^{''}, e_9, e_{10}\}$

$SW_{list} = \{e_1, e_3, e_7^{'}, e_6^{'}\}$

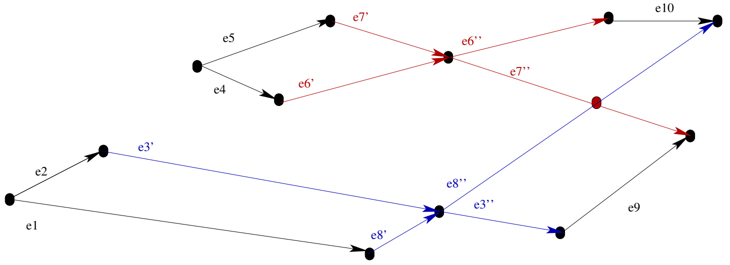
Detect $e_8 \cap e_3$:

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_3^{''}, e_8^{''}, e_7^{''}, e_6^{''}, e_9, e_{10}\}$

$SW_{list} = \{e_3^{'}, e_8^{'}, e_7^{'}, e_6^{'}\}$

Intermediate step

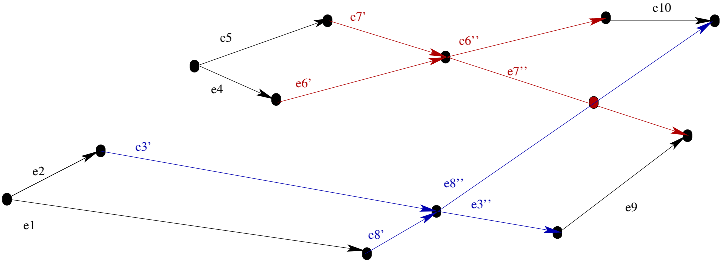
Refinements of the algorithm (next):



$SW_{list} = \{e_1, e_3, e_7^{'}, e_6^{'}\}$
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_7^{''}, e_6^{''}, e_9, e_{10}\}$
$SW_{list} = \{e_8, e_3, e_7^{'}, e_6^{'}\}$
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_3^{''}, e_8^{''}, e_7^{''}, e_6^{''}, e_9, e_{10}\}$
$SW_{list} = \{e_3^{'}, e_8^{'}, e_7^{'}, e_6^{'}\}$

Intermediate step

Refinements of the algorithm (next):



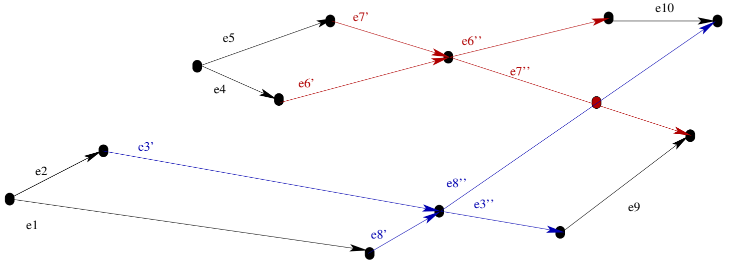
$SW_{list} = \{e_3', e_8', e_7', e_6'\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_3'', e_8'', e_7'', e_6'', e_9, e_{10}\}$

$SW_{list} = \{e_3'', e_8', e_7', e_6'\}$

Intermediate step

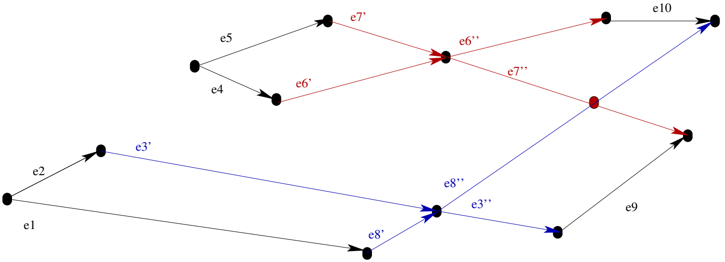
Refinements of the algorithm (next):



$SW_{list} = \{e_3'', e_8', e_7', e_6'\}$
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_3'', e_8'', e_7'', e_6'', e_9, e_{10}\}$
$SW_{list} = \{e_3'', , e_8'', e_7', e_6'\}$

Intermediate step

Refinements of the algorithm (next):



$SW_{list} = \{e_3^{''}, e_8^{'}, e_7^{'}, e_6^{'}\}$
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_3^{''}, e_8^{''}, e_7^{''}, e_6^{''}, e_9, e_{10}\}$
$SW_{list} = \{e_3^{''}, , e_8^{''}, e_7^{''}, e_6^{'}\} \Rightarrow$
$I = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$
$E_I = \{(e_6, e_7), (e_8, e_3), (e_8^{''}, e_7^{''})\}$

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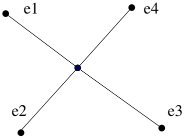
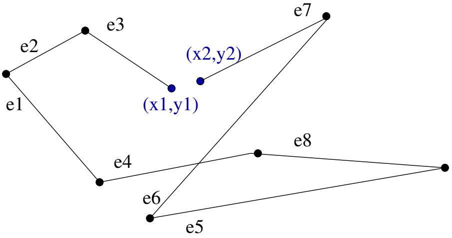
Intermediate step

Degenerate cases in the algorithm (unsolved yet!):

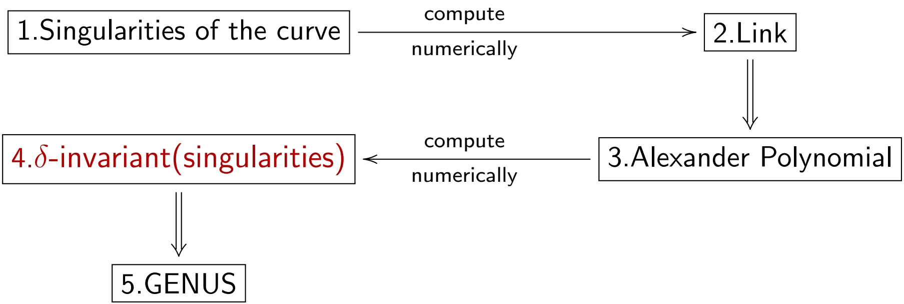
Case 1: Find condition s.t. $I = e_1 \cap e_2 \cap e_3 \cap e4$



Case 2: Find $\epsilon > 0$ s.t. $(x1 - x2)^2 + (y1 - y2)^2 < \epsilon^2$



Next


1.Singularities of the curve
compute
numerically
2.Link
3.Alexander Polynomial
compute
numerically
4.δ-invariant(singularities)
5.GENUS

Computing the δ-invariant of the singularity

- Input:
  - $C \subset \mathbb{C}^2$ complex algebraic curve;
  - $z \in Singularities(C)$;
  - $\Delta(t_1, .., t_p)$ - Alexander polynomial of z;
  - $r =$ number of variables in Δ (branches of C through z);
  - $\mu =$ degree of Δ (multiplicity of z);
- Output:
  - $\delta_z > 0$ s.t.

δ_z is an invariant that measures
the number of double points of C at z.

Computing the δ-invariant of the singularity

- Method: based on Milnor's research on singularities of complex hypersurfaces.

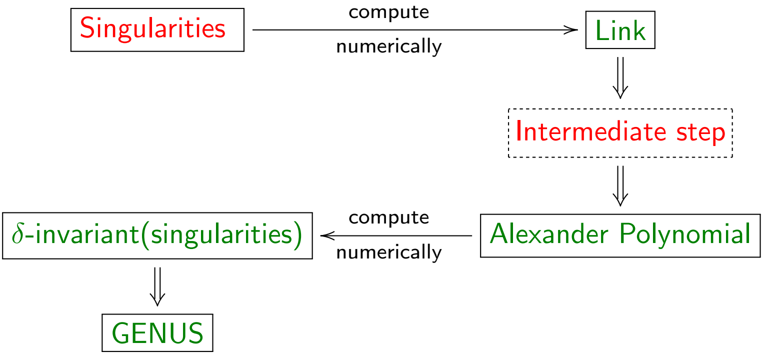
$$\Delta(t_1, ..., t_p) : r, \mu$$

$$r \geq 2 \Rightarrow \delta = \frac{1}{2}(\mu + r)$$

$$r = 1 \Rightarrow \delta = \frac{1}{2}\mu$$

Summary

- At present: for symbolic coefficients



- Future work: tests for algorithm with numeric coefficients

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Conclusion

- first results and test experiments were presented;
- Future work:
  - deeper introspection into some mathematical aspects (i.e. Milnor's fibration, Alexander polynomial);
  - correctness/completeness for the algorithm;
  - implementation of the algorithm;
  - analysis of the algorithm.

Thank you for your attention.