A symbolic-numeric algorithm for genus computation

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Numerical Analysis and Symbolic Computation

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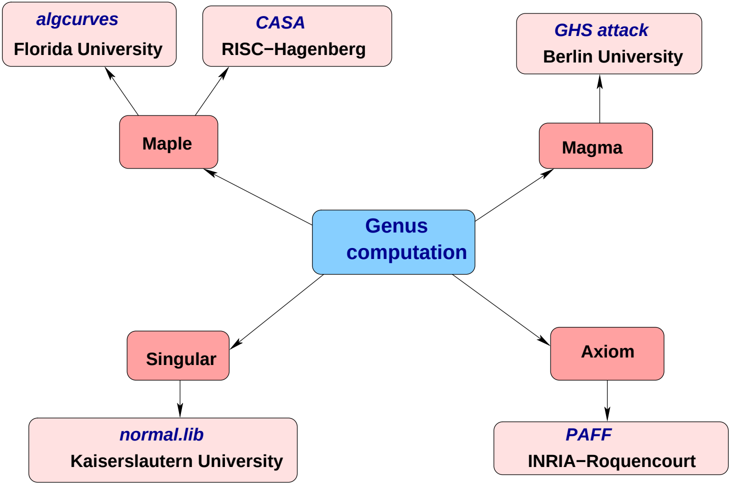
Symbolic Algorithms:



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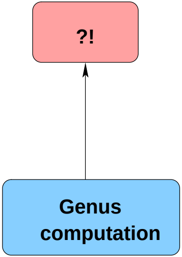
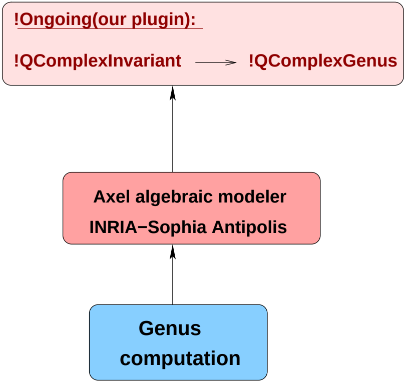
Numeric Algorithms:



Symbolic-Numeric Algorithms:

DK Project: Symbolic-Numeric techniques for genus computation and parametrization (project leader: Prof. Dr. Josef Schicho).



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What?

- Input:
  - $\mathbb{C}$ field of complex numbers;
  - $F \in \mathbb{C}[z, w]$ irreducible with coefficients of limited accuracy [1];
  - $C = \{(z, w) \in \mathbb{C}^2 | F(z, w) = 0\} =$
    $= \{(x, y, u, v) \in \mathbb{R}^4 | F(x + iy, u + iv) = 0\}$ complex algebraic curve (d is the degree, $Sing(C)$ is the set of singularities);
- Output:
  - approximate $genus(C)$ s.t.

$$genus(C) = \frac{1}{2}(d-1)(d-2) - \sum_{P \in Sing(C)} \delta\text{-invariant}(P);$$

[1] For now: symbolic coefficients

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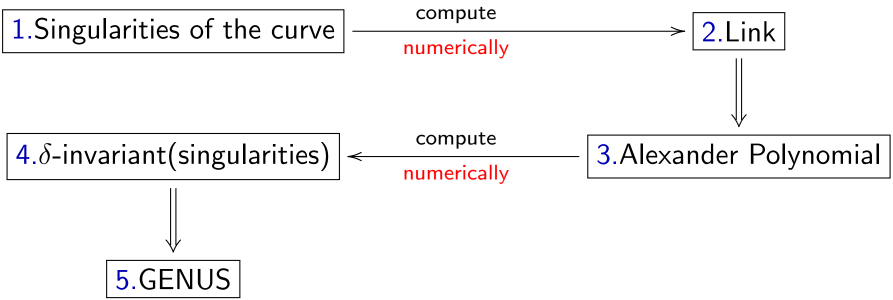
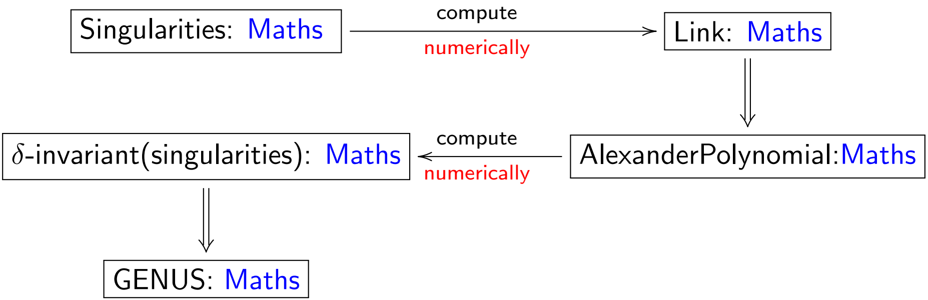
How?

- Strategy for computing the genus



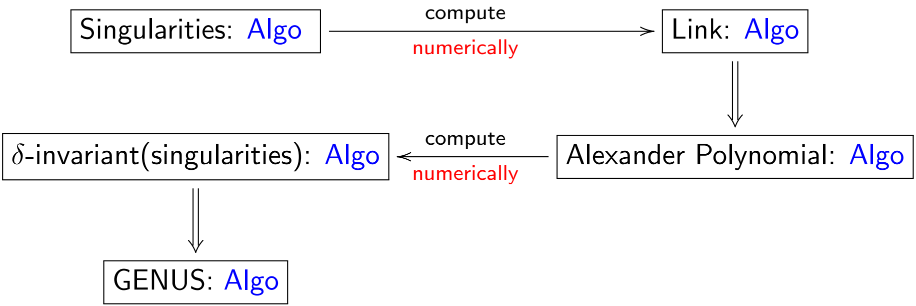
How?

- Method for computing the genus



How?

- Algorithm for the method



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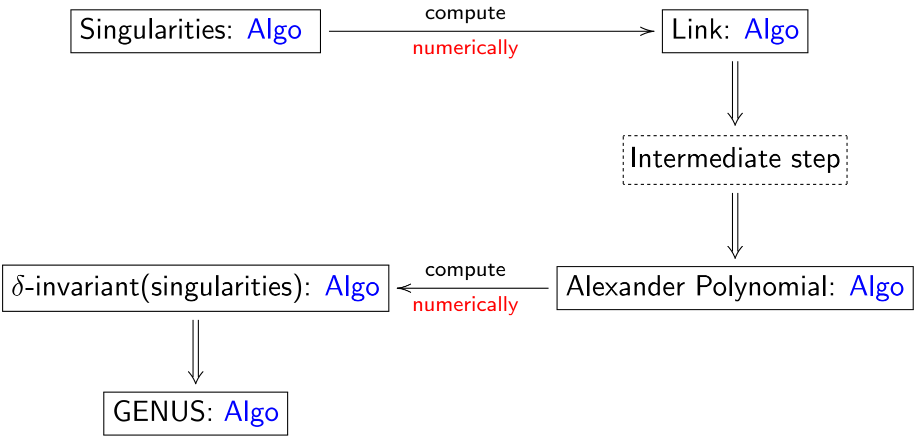
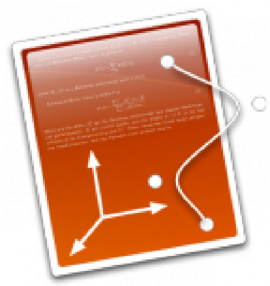


Solving the problem

Implementation of the algorithm

- *Axel* algebraic geometric modeler [a]
  - developed by *Galaad* team (INRIA Sophia-Antipolis);
  - written in Qt scripting language;
  - provides algebraic tools for:
    - implicit curves;
    - implicit surfaces.



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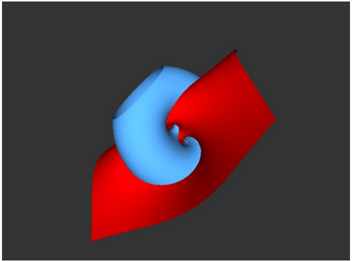

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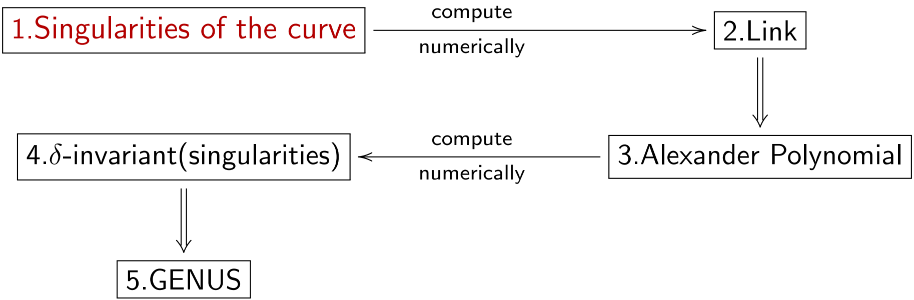
First


1.Singularities of the curve
compute
numerically
2.Link
3.Alexander Polynomial
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4.δ-invariant(singularities)
5.GENUS

Computing the singularities of the curve

- Input:
  - $F \in \mathbb{C}[z, w]$
  - $C = \{(z, w) \in \mathbb{C}^2 | F(z, w) = 0\}$
- Output:
  - $Sing(C) = \{(z_0, w_0) \in \mathbb{C}^2 | F(z_0, w_0) = 0, \frac{\delta F}{\delta z}(z_0, w_0) = 0, \frac{\delta F}{\delta w}(z_0, w_0) = 0\}$

Method: $\Rightarrow$ solve overdeterminate system of polynomial equations in $\mathbb{C}^2$:

$$\begin{cases} F(z_0, w_0) = 0 \\ \frac{\delta F}{\delta z}(z_0, w_0) = 0 \\ \frac{\delta F}{\delta w}(z_0, w_0) = 0 \end{cases}, \tag{1}$$

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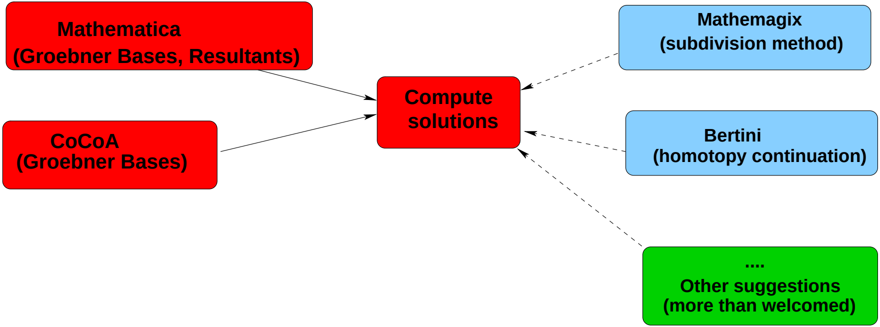
Computing the singularities of the curve

or in $\mathbb{R}^4$: $F(z,w) = F(x+iy, u+iv) = s(x,y,u,v) + it(x,y,u,v)$

$$\begin{cases} s(x_0,y_0,u_0,v_0) = 0 \\ t(x_0,y_0,u_0,v_0) = 0 \\ \\ \dfrac{\delta s}{\delta x}(x_0,y_0,u_0,v_0) = 0 \\ \\ \dfrac{\delta t}{\delta x}(x_0,y_0,u_0,v_0) = 0 \\ \\ \dfrac{\delta s}{\delta u}(x_0,y_0,u_0,v_0) = 0 \\ \\ \dfrac{\delta t}{\delta u}(x_0,y_0,u_0,v_0) = 0 \end{cases} , \qquad (2)$$

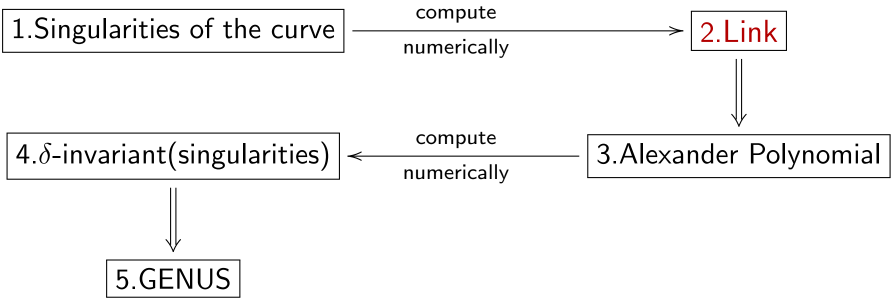
Computing the singularities of the curve

Using numeric input polynomials



Note: so far an open problem.

Next



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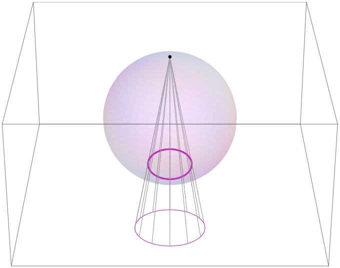
Computing the link of the singularity

- Why the link of a singularity?
  - helps in understanding the topology of a complex curve near a singularity;
- How do we compute the link?
  - use stereographic projection;



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Computing the link of the singularity

Method (based on Milnor's results)

1. Let $C = \{(x,y,u,v) \in \mathbb{R}^4 | F(x,y,u,v) = 0\}$ s.t. $(0,0,0,0) \in Sing(C)$

2. Consider $S_{(0,\epsilon)} := S = \{(x,y,u,v) \in \mathbb{R}^4 | x^2 + y^2 + u^2 + w^2 = \epsilon^2\}$,
   $X = C \bigcap S_{(0,\epsilon)} \subset \mathbb{R}^4$

3. For $P \in S \setminus C$ take $f : S \setminus \{P\} \to \mathbb{R}^3, f(x,y,u,v) = (\frac{x}{\epsilon - v}, \frac{y}{\epsilon - v}, \frac{u}{\epsilon - v})$,
   $f^{-1} : \mathbb{R}^3 \to S \setminus \{P\}$
   $f^{-1}(a,b,c) = (\frac{2a\epsilon}{1+a^2+b^2+c^2}, \frac{2b\epsilon}{1+a^2+b^2+c^2}, \frac{2c\epsilon}{1+a^2+b^2+c^2}, \frac{\epsilon(a^2+b^2+c^2-1)}{1+a^2+b^2+c^2})$

4. Compute $f(X) = \{(a,b,c,) \in \mathbb{R}^3 | F(...) = 0\} \Leftrightarrow$
   $f(X) = \{(a,b,c,) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$ and
   B for $f(X) = \{(a,b,c,) \in B \subset \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$
   For small ϵ, $f(X)$ is a link

Note: A link is a closed loop in $\mathbb{R}^3$ that does not intersect itself.

Computing the link of the singularity

Why Axel?

It computes numerically the topology of implicit curves in $\mathbb{R}^3$

- For $C^4 = \{(z,w) \in \mathbb{C}^2 | z^3 - w^2 = 0\} \subset \mathbb{R}^4$ get
- $f(C^4 \cap S) := C =$
  $= \{(a,b,c) \in \mathbb{R}^3 | ReF(...) = 0, ImF(...) = 0\}$
- compute $Graph(C) = \langle \mathcal{V}, \mathcal{E} \rangle$ with
  $\mathcal{V} = \{p = (m,n,q) \in \mathbb{R}^3\}$
  $\mathcal{E} = \{(i,j) | i,j \in \mathcal{V}\}$
- s.t. $Graph(C) \cong_{isotopic} C$

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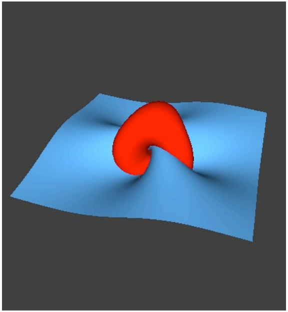

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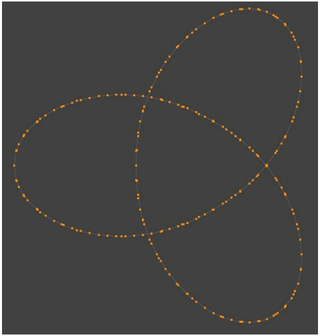

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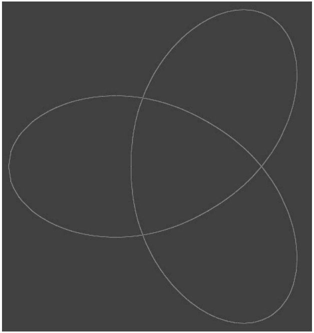

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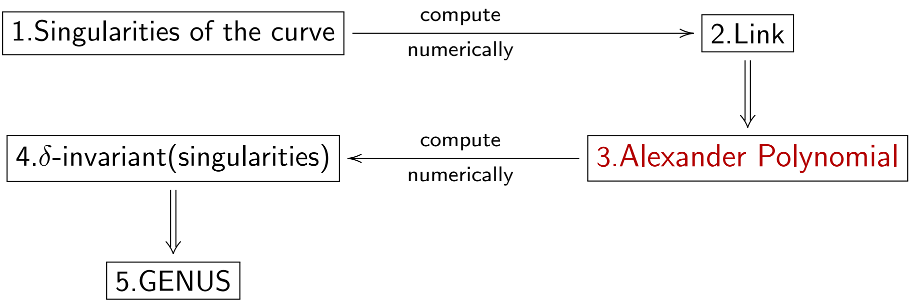
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Preliminaries

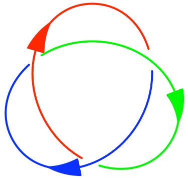
A double point of a projection is called a **crossing point**.

A **diagram** is the image under regular projection, together with the information on each crossing telling which branch goes over and which under.

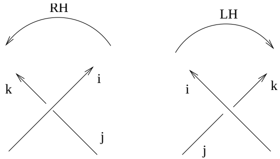
An **arc** is the part of a diagram between two undercrossings.

A crossing is:
-**righthanded** if the underpass traffic goes from right to left.
-**lefthanded** if the underpass traffic goes from left to right.

Diagram and arcs



Crossings



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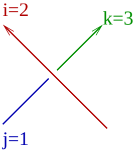
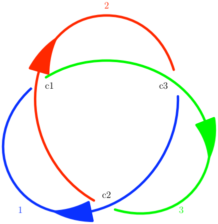
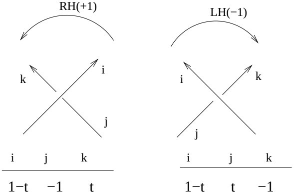
Computing the Alexander polynomial of the link





$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_1 & -1 & 2 & 1 & 3 \\ & & & & \end{array}\right)$$

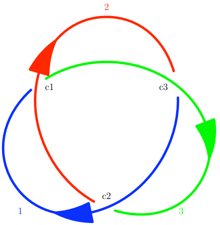
$$P(L) = \left(\quad \right)$$

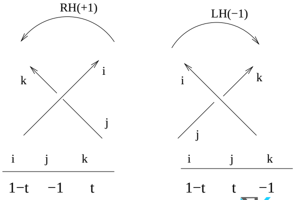


Computing the Alexander polynomial of the link

$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_1 & -1 & 2 & 1 & 3 \\ & & 1-t & t & -1 \\ & & & & \end{array}\right)$$

$$P(L) = \begin{pmatrix} 2 & 1 & 3 \\ 1-t & t & -1 \\ & & \end{pmatrix}$$





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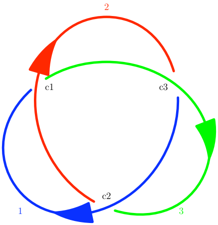
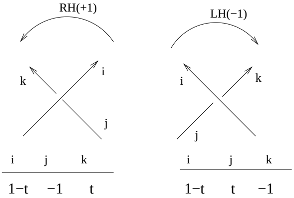
$$P(L) = \left(\begin{array}{ccc} 1 & 2 & 3 \\ t & 1-t & -1 \\ & & \end{array} \right)$$





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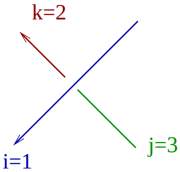

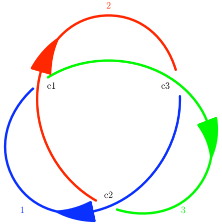

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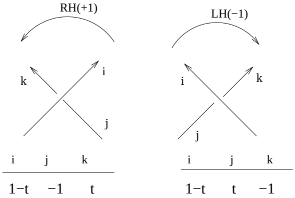

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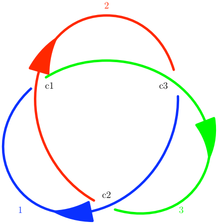

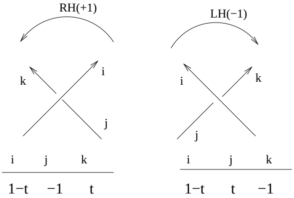

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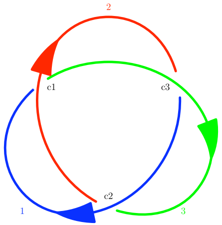

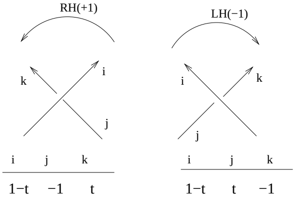

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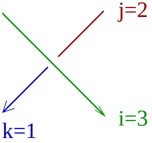

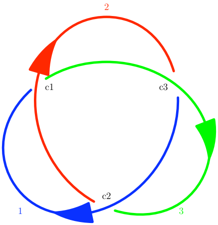

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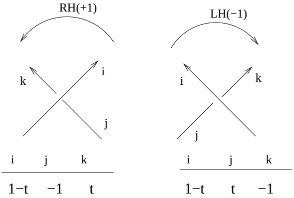

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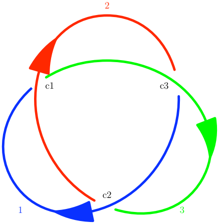
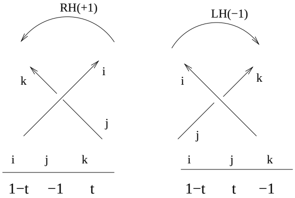
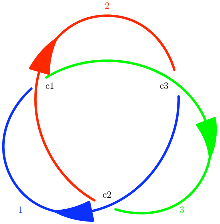


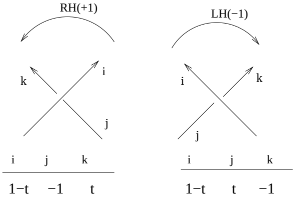


Computing the Alexander polynomial of the link

$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline & & & & \\ c_3 & -1 & 3 & 2 & 1 \\ & & 1-t & t & -1 \end{array}\right)$$

$$P(L) = \left(\begin{array}{ccc} & & \\ 1 & 2 & 3 \\ -1 & t & 1-t \end{array}\right)$$





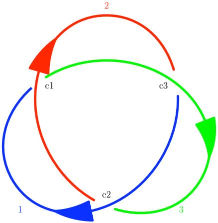
Computing the Alexander polynomial of the link

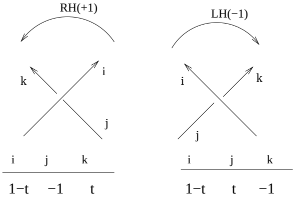
$$M(L) = \left(\begin{array}{c|cccc} & type & label_i & label_j & label_k \\ \hline c_1 & -1 & 2 & 1 & 3 \\ c_2 & -1 & 1 & 3 & 2 \\ c_3 & -1 & 3 & 2 & 1 \end{array}\right)$$

$$P(L) = \begin{pmatrix} t & 1-t & -1 \\ 1-t & -1 & t \\ -1 & t & 1-t \end{pmatrix}$$

$$det(P(L)) = -t^2 + t - 1$$

$$\Delta(L) := \Delta(t) = Normalise(det(P(L))) = t^2 - t + 1$$





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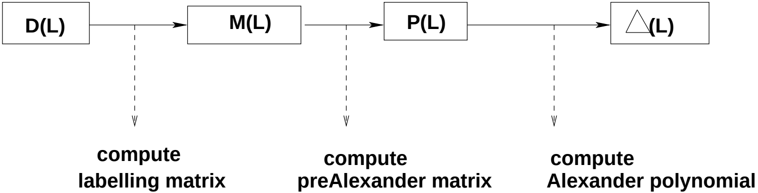
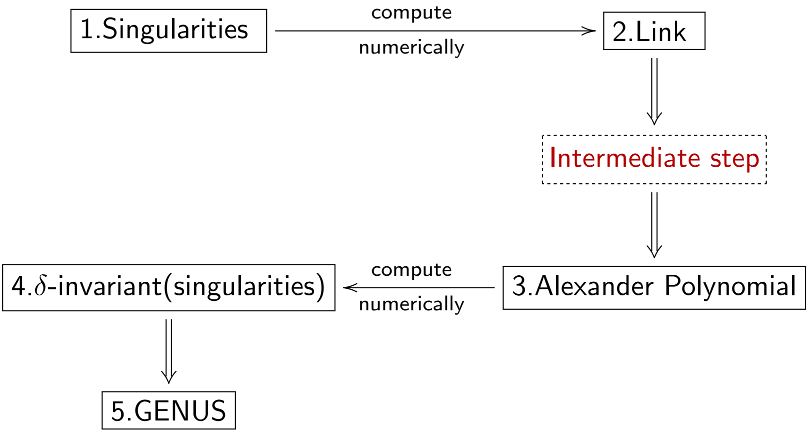
Computing the Alexander polynomial of the link

- Input:
  - $L = K_1 \cup ... \cup K_m$ with n - crossings
  - $D(L)$- oriented diagram of L
- Output:
  - $\Delta_L(t_1, ...t_m) \in \mathbb{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$
- Method: consists of several steps

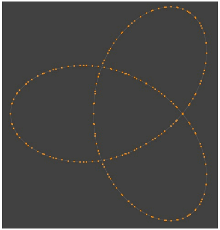


- Need D(L)!

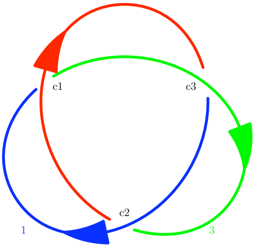
Next


1.Singularities
compute
numerically
2.Link
Intermediate step
3.Alexander Polynomial
compute
numerically
4.δ-invariant(singularities)
5.GENUS

Intermediate step

- 

$\overset{?}{\Rightarrow}$



- $G(L) = \langle P, E \rangle$

$D(L)$

- p(index,x,y,z)

e(pointS, pointD)

$\longrightarrow$ number of arcs, crossings
$\longrightarrow$ type of crossings (under, over)
$\longrightarrow$ number of knots in the link(orientation)

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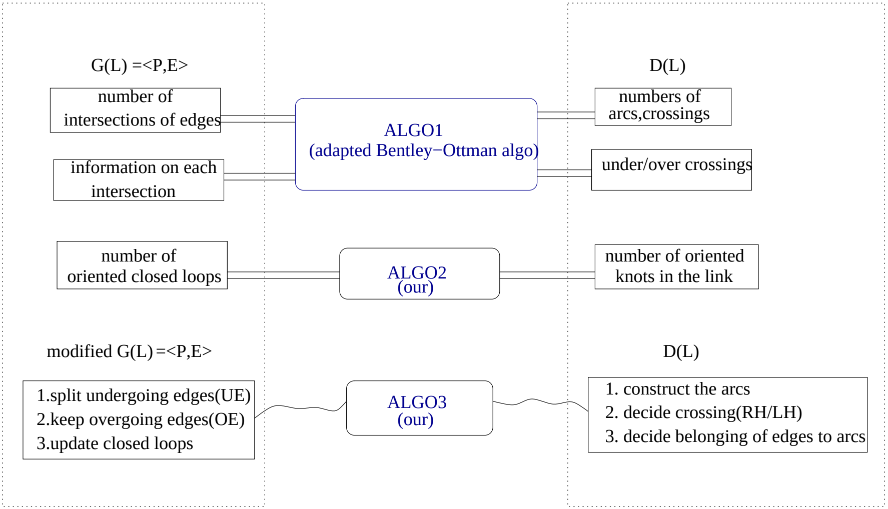
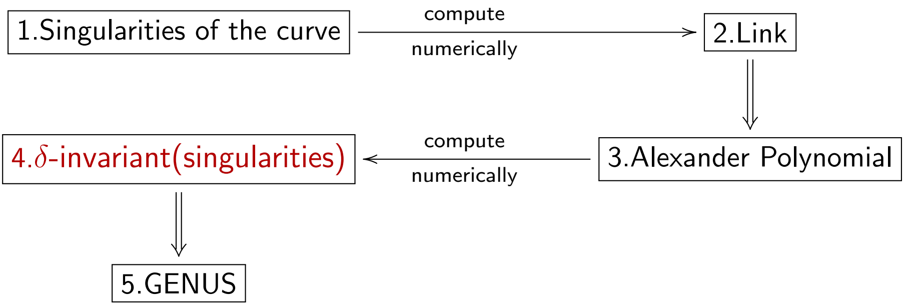
Intermediate step



Next


1.Singularities of the curve
compute
numerically
2.Link
3.Alexander Polynomial
compute
numerically
4.δ-invariant(singularities)
5.GENUS

Computing the δ-invariant of the singularity

- Input:
  - $C \subset \mathbb{C}^2$ complex algebraic curve;
  - $z \in Singularities(C)$;
  - $\Delta(t_1, .., t_p)$ - Alexander polynomial of z;
  - $r =$ number of variables in Δ (branches of C through z);
  - $\mu =$ degree of Δ (multiplicity of z);
- Output:
  - $\delta_z > 0$ s.t. δ_z measures the number of double points of C at z.
- Method (based on Milnor's research)

$$\Delta(t_1, ..., t_p) : r, \mu$$

$$r \geq 2 \Rightarrow \delta = \frac{1}{2}(\mu + r)$$

$$r = 1 \Rightarrow \delta = \frac{1}{2}\mu$$

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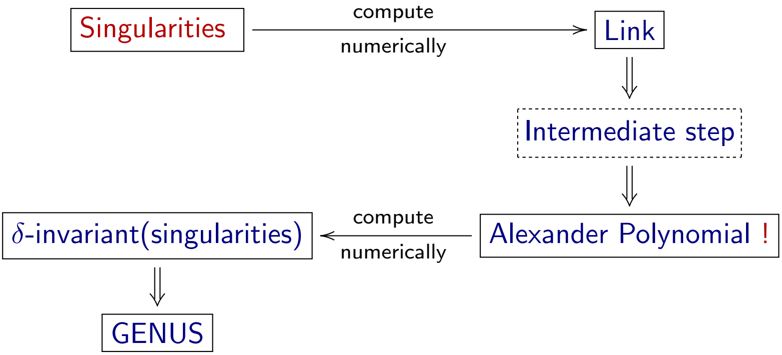
Summary

- Present work: for symbolic coefficients



- Future work: tests for algorithm with numeric coefficients

1. Motivation

2. Describing the problem
   - What?

3. Solving the problem
   - How?

4. Current results

5. Conclusion and future work

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Conclusion

- first results (methods and algorithms) were presented;
- Future work:
  - deeper introspection into some mathematical aspects (i.e. numeric computation);
  - complete implementation of the algorithm;
  - correctness proof for the algorithm;
  - analysis of the algorithm.

Thank you for your attention.
Questions?